Euclid's "Elements" Redux

Daniel Callahan

John Casey

Sir Thomas Heath

Version 2016-169

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“Don’t just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?”

- Paul Halmos

“Pure mathematics is, in its way, the poetry of logical ideas.”

- Albert Einstein

“Math is like going to the gym for your brain. It sharpens your mind.”

- Danica McKellar

“One of the endlessly alluring aspects of mathematics is that its thorniest paradoxes have a way of blooming into beautiful theories.”

- Philip J. Davis

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CHAPTER 0

About this project

The goal of this textbook is to provide an edition of Euclid's "Elements" that is easy to read, inexpensive, has an open culture license, and can legally be distributed anywhere in the world.

The title *Euclid's "Elements" Redux* indicates that while this edition covers each of Euclid's results, the proofs themselves have been written using modern mathematics. This is because mathematics has almost completely changed since Euclid wrote "The Elements".

In Euclid's Hellenistic culture, a number was synonymous with a length that could be measured with a ruler, a rope, or some other such device. This gave Euclid access to all of the positive numbers but none of the negative numbers. (The Greeks also had a difficult time working with 0.) But since the development of algebra wouldn't begin for another 1,000 years, Euclid could not have known that the result of Book I Proposition 47 could be written as $a^2 + b^2 = c^2$.

This means that when a modern student encounters Euclid for the first time, he or she has two problems to solve: the logic of the proof, and Euclid's now archaic concepts of what numbers are and how they related to each other. While Euclid's original intentions are of great interest to math historians, they can confuse beginners.

This edition helps students understand Euclid's results by using modern proofs, and these results contain material that every math student should know. This is one reason Euclid's "Elements" was the world's most important mathematics textbook for about 2,200 years.

Let that number sink in for a moment... one math textbook was used by much of the literate world for over two thousand years. Why should this be? Lack of competition? At certain times and places, yes, but in schools where Euclid's "Elements" got a foothold, other textbooks soon followed. Therefore, lack of competition can't be the complete answer.

Elegant proofs? Most of Euclid's proofs can and were rewritten as new math was developed. So while the originals remain a model for how proofs can be written, they aren't irreplaceable.

Content? Until the development of algebra and calculus, Euclid's "Elements" covered everything a novice mathematician needed to know. And even after calculus was developed, "The Elements" still had its place. Isaac Newton's assistant claimed that in five years of service, he heard Newton laugh just once: a student asked if Euclid were still worth studying, and Newton laughed at his ignorance. But most students only studied the first six Books (i.e., the first six of the thirteen scrolls that made up the original edition of "The Elements"). Therefore not everyone studied "The Elements" exclusively for its content.

So, whatever is valuable about "The Elements" must be present in the first half of the book; but each Book begins with a secure intellectual foundation (definitions and axioms) and builds upwards one proposition at a time.

And that, I think, is the key to answering this question: not only does "The Elements" help a student learn geometry, but it also trains the student's mind to use logic properly. Learning algebra helps a student to learn calculus and statistics, but learning geometric proofs helps a student to think clearly about politics, art, music, design, coding, law... any subject that requires rational thought can be better understood after working through "The Elements".

The reason "The Elements" was needed in the past, and why it's needed today, is that it helps students learn to think clearly.

But if it's such a good textbook, why was it abandoned after the 19th century?

The last quarter of that decade saw an explosion of new mathematics, including the construction of formal logic and a new rigorous approach to proof writing. Suddenly, Euclid began to look dated; and as calculus became the gateway to engineering and the sciences, algebra became the course every student seemed to need. Geometric proofs became a luxury rather than a necessity. This was a mistake which has hindered generations from learning to think logically.

While the mathematical developments of the late 19th century led to a mathematical Golden Age (one which we are still in), one of the best pedagogical tools ever written was discarded. Perhaps if "The Elements" were rewritten to conform to current textbook standards, its importance in geometry, proof writing, and as a case-study in logic might once again be recognized by the worldwide educational community.

This edition of Euclid can be used in a number of ways:

- As a means of introducing students to proofs and to writing proofs.
- As a supplement to any geometry or analytical geometry textbook.
- As a textbook for self-study and homeschooling.

This edition also contains homework problems and a partial answer key.

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This textbook requires its student to work slowly and carefully through each section. The student should check every result stated in the book and not take anything on faith. While this process may seem tedious, it is exactly this attention to detail which separates those who understand mathematics from those who do not.

The prerequisites for this textbook include a desire to solve problems and to learn mathematical logic. Some background in proportions, algebra, trigonometry, and possibly linear algebra may be helpful. However, there is no mistaking the fact that the homework in this textbook is unlike that of most other elementary math textbooks. Rather than posing questions which require calculations to answer, these problems require a written proof. Students who have no knowledge of proof writing should only be asked to solve the simplest of problems at first; students or parents of students who have no idea how to proceed are encouraged to read Richard Hammack's Book of Proof, 2nd ed. or a similar proof-writing textbook (see section 1.5 for details). But it is vitally important to understand that to mathematicians **proofs are mathematics**. Most mathematicians at the Ph.D. level leave the calculation of difficult problems to physicists and engineers; the job of a mathematician is to prove that an answer to the problem exists before the calculations begin. So, despite the fact that most students take classes described as "math" or "maths", most people (including those who claim to hate math) do not know what mathematics actually is; in essence, math is proofs.

The figures in this textbook were created in GeoGebra. They can be found in the images folder in the source files for this textbook. Files with extensions .ggb are GeoGebra files, and files with the .eps extension are graphics files. Instructional videos are also available on YouTube which demonstrate how to use GeoGebra: `http://www.youtube.com/channel/UCjrVV46Fijv-Pi5VcFm3dCQ`

This document was composed using:

Debian Linux	http://www.debian.org/
GeoGebra	http://www.geogebra.org/
Linux Mint	http://www.linuxmint.com/
LyX	http://www.lyx.org/
Windows 7	http://windows.microsoft.com/
Xubuntu Linux	http://xubuntu.org/

0.1. Contributions & Acknowledgments

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- Deirdre Callahan (help with Raphael's "The School of Athens")
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Daniel Callahan would like to thank Dr. Wally Axmann[1], Dr. Elizabeth Behrman[2], Karl Elder[3], Dr. Thalia Jeffres[4], Dr. Kirk Lancaster[5], Dr. Phil Parker[6], Dr. Gregory B. Sadler[7], Valorie Starr, and the good people at Weatherford College[8].

0.2. Dedication

This book is dedicated to everyone in the educational community who believes that algebra provides a better introduction to mathematics than geometry.

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[1] http://www.math.wichita.edu/~axmann/
[2] http://webs.wichita.edu/physics/behrman/behr.htm
[3] http://karlelder.com
[4] http://www.math.wichita.edu/~jeffres/
[5] http://kirk.math.wichita.edu/
[6] http://www.math.wichita.edu/~pparker/
[7] http://gregorybsadler.com/
[8] http://www.wc.edu

0.3. About Euclid's "Elements"

[This chapter has been adapted from an entry in Wikipedia.[9]]

Euclid's "Elements" is a mathematical and geometric treatise consisting of 13 books written by the ancient Greek mathematician Euclid in Alexandria, Egypt, c.300 BC. It is a collection of definitions, postulates (axioms), propositions (theorems and constructions), and mathematical proofs of the propositions. The thirteen books cover Euclidean geometry and the ancient Greek version of elementary number theory. The work also includes an algebraic system that has become known as geometric algebra, which is powerful enough to solve many algebraic problems, including the problem of finding the square root of a number. With the exception of Autolycus' "On the Moving Sphere", the Elements is one of the oldest extant Greek mathematical treatises, and it is the oldest extant axiomatic deductive treatment of mathematics. It has proven instrumental in the development of logic and modern science. The name "Elements" comes from the plural of "element". According to Proclus, the term was used to describe a theorem that is all-pervading and helps furnish proofs of many other theorems. The word "element" is in the Greek language the same as "letter": this suggests that theorems in the "Elements" should be seen as standing in the same relation to geometry as letters to language. Later commentators give a slightly different meaning to the term "element", emphasizing how the propositions have progressed in small steps and continued to build on previous propositions in a well-defined order.

[9] `http://en.wikipedia.org/wiki/Euclid's_Elements`
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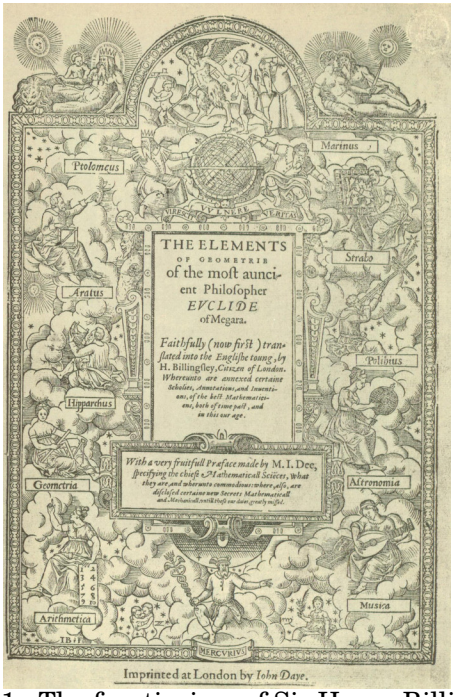
Ptolomeus
Marinus
Strabo
Arattus
Polibius
Hipparchus
Geometria
Astronomia
Musica
Arithmetica
THE ELEMENTS
OF GEOMETRIE
of the most auncient Philosopher
EVCLIDE
of Megara.
Faithfully (now first) translated into the Englishe toung, by H. Billingsley, Citizen of London.
With a very fruitfull Preface made by M. I. Dee, specifying the chiefe Mathematicall Sciences, what they are, and wherunto commodious: where, also, are declared certaine new Secrets Mathematicall and Mechanicall, until this our time, greatly missed.
MERCVRIVS
Imprinted at London by Iohn Daye.

FIGURE 0.3.1. The frontispiece of Sir Henry Billingsley's first English version of Euclid's Elements, 1570.

Euclid's "Elements" has been referred to as the most successful and influential textbook ever written. Being first set in type in Venice in 1482, it is one of the very earliest mathematical works to be printed after the invention of the printing press and was estimated by Carl Benjamin Boyer to be second only to the Bible in the number of editions published (the number reaching well over one thousand). For centuries, when the *quadrivium*[10] was included in the curriculum of all university students, knowledge of at least part of Euclid's "Elements" was required of all students. Not until the 20th century, by which time its content was universally taught through other school textbooks, did it cease to be considered something that all educated people had read.

0.3.1. History.

[10]In Medieval Europe, the quadrivium were the four subjects (arithmetic, geometry, music, and astronomy) taught after teaching the trivium. Together, the trivium (grammar, rhetoric, logic) and the quadrivium comprised the seven liberal arts (based on thinking skills), as opposed to the practical arts (such as medicine and architecture).
See also: `https://en.wikipedia.org/wiki/Quadrivium`

0.3.1.1. *Basis in earlier work.* Scholars believe that the Elements is largely a collection of theorems proved by other mathematicians supplemented by some original work. Proclus, a Greek mathematician who lived several centuries after Euclid, wrote in his commentary: "Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefutable demonstration the things which were only somewhat loosely proved by his predecessors". Pythagoras was probably the source of most of books I and II,[11] Hippocrates of Chios of book III, and Eudoxus book V, while books IV, VI, XI, and XII probably came from other Pythagorean or Athenian mathematicians. Euclid often replaced fallacious proofs with his own more rigorous versions. The use of definitions, postulates, and axioms dated back to Plato. The "Elements" may have been based on an earlier textbook by Hippocrates of Chios, who also may have originated the use of letters to refer to figures.

0.3.1.2. *Transmission of the text.* In the fourth century AD, Theon of Alexandria extended an edition of Euclid which was so widely used that it became the only surviving source until François Peyrard's 1808 discovery at the Vatican of a manuscript not derived from Theon's. This manuscript, the Heiberg manuscript, is from a Byzantine workshop c. 900 and is the basis of modern editions. Papyrus Oxyrhynchus 29 is a tiny fragment of an even older manuscript, but only contains the statement of one proposition.

Although known to Cicero, there is no extant record of the text having been translated into Latin prior to Boethius in the fifth or sixth century. The Arabs received the Elements from the Byzantines in approximately 760; this version, by a pupil of Euclid called Proclo, was translated into Arabic under Harun al Rashid c.800. The Byzantine scholar Arethas commissioned the copying of one of the extant Greek manuscripts of Euclid in the late ninth century. Although known in Byzantium, the "Elements" was lost to Western Europe until c. 1120, when the English monk Adelard of Bath translated it into Latin from an Arabic translation.

The first printed edition appeared in 1482 (based on Campanus of Novara's 1260 edition), and since then it has been translated into many languages and published in about a thousand different editions. Theon's Greek edition was

[11]Roman numerals, the numeric system used in ancient Rome and throughout Europe until the beginning of the Middle Ages, use combinations of letters from the Latin alphabet to signify values. The numbers 1 to 10 can be expressed in Roman numerals I, II, III, IV, V, VI, VII, VIII, IX, X.
See also: `https://en.wikipedia.org/wiki/Roman_numerals`

recovered in 1533. In 1570, John Dee provided a widely respected "Mathematical Preface", along with copious notes and supplementary material, to the first English edition by Henry Billingsley.

Copies of the Greek text still exist, some of which can be found in the Vatican Library and the Bodleian Library in Oxford. The manuscripts available are of variable quality and are invariably incomplete. By careful analysis of the translations and originals, hypotheses have been made about the contents of the original text (copies of which are no longer available).

Ancient texts which refer to the "Elements" itself and to other mathematical theories that were current at the time it was written are also important in this process. Such analyses are conducted by J.L. Heiberg and Sir Thomas Little Heath in their editions of the text.

Also of importance are the scholia, or annotations, to the text. These additions which often distinguished themselves from the main text (depending on the manuscript) gradually accumulated over time as opinions varied upon what was worthy of explanation or further study.

0.3.2. Influence. The "Elements" is still considered a masterpiece in the application of logic to mathematics. In historical context, it has proven enormously influential in many areas of science. Scientists Nicolaus Copernicus, Johannes Kepler, Galileo Galilei, and Sir Isaac Newton were all influenced by the Elements, and applied their knowledge of it to their work. Mathematicians and philosophers, such as Bertrand Russell, Alfred North Whitehead, and Baruch Spinoza, have attempted to create their own foundational "Elements" for their respective disciplines by adopting the axiomatized deductive structures that Euclid's work introduced.

The austere beauty of Euclidean geometry has been seen by many in western culture as a glimpse of an otherworldly system of perfection and certainty. Abraham Lincoln kept a copy of Euclid in his saddlebag, and studied it late at night by lamplight; he related that he said to himself, "You never can make a lawyer if you do not understand what demonstrate means; and I left my situation in Springfield, went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight." Edna St. Vincent Millay wrote in her sonnet Euclid Alone Has Looked on Beauty Bare, "O blinding hour, O holy, terrible day, When first the shaft into his vision shone Of light anatomized!" Einstein recalled a copy of the "Elements" and

a magnetic compass as two gifts that had a great influence on him as a boy, referring to the Euclid as the "holy little geometry book".

FIGURE 0.3.2. The Italian Jesuit Matteo Ricci (left) and the Chinese mathematician Xu Guangqi (right) published the Chinese edition of Euclid's "Elements" in 1607.

The success of the "Elements" is due primarily to its logical presentation of most of the mathematical knowledge available to Euclid. Much of the material is not original to him, although many of the proofs are his. However, Euclid's systematic development of his subject, from a small set of axioms to deep results, and the consistency of his approach throughout the Elements, encouraged its use as a textbook for about 2,000 years. The "Elements" still influences modern geometry books. Further, its logical axiomatic approach and rigorous proofs remain the cornerstone of mathematics.

0.3.3. Euclid's method and style of presentation. Euclid's axiomatic approach and constructive methods were widely influential.

As was common in ancient mathematical texts, when a proposition needed proof in several different cases, Euclid often proved only one of them (often the most difficult), leaving the others to the reader. Later editors such as Theon often interpolated their own proofs of these cases.

Euclid's presentation was limited by the mathematical ideas and notations in common currency in his era, and this causes the treatment to seem awkward to the modern reader in some places. For example, there was no notion of an angle greater than two right angles, the number 1 was sometimes treated separately from other positive integers, and multiplication was treated geometrically; in fact, he did not use the product of more than three different numbers. The geometrical treatment of number theory may have been because the alternative would have been the extremely awkward Alexandrian system of numerals. [12]

The presentation of each result is given in a stylized form, which, although not invented by Euclid, is recognized as typically classical. It has six different parts: first is the statement of the proposition in general terms (also called the enunciation). Then the setting-out, which gives the figure and denotes particular geometrical objects by letters. Next comes the definition or specification which restates the enunciation in terms of the particular figure. Then the construction or machinery follows. It is here that the original figure is extended to follow the proof. The proof itself follows. Finally, the conclusion connects the proof to the enunciation by stating the specific conclusions constructed in the proof in the general terms of the enunciation.

No indication is given of the method of reasoning that led to the result, although the data does provide instruction about how to approach the types of problems encountered in the first four books of the Elements. Some scholars have tried to find fault in Euclid's use of figures in his proofs, accusing him of writing proofs that depended on the specific figures constructed rather than the general underlying logic (especially concerning Proposition II of Book I). However, Euclid's original proof of this proposition is general, valid, and does not depend on the figure used as an example to illustrate one given configuration.

0.3.3.1. *Criticism.* While Euclid's list of axioms in the "Elements" is not exhaustive, it represents the most important principles. His proofs often invoke axiomatic notions which were not originally presented in his list of axioms. Later editors have interpolated Euclid's implicit axiomatic assumptions in the list of formal axioms.

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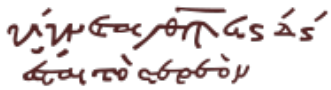
[12] As one example, the top line is the number $9996\frac{4}{6}$ written in Alexandrian numerals:



See also: https://en.wikipedia.org/wiki/Greek_numerals

For example, in the first construction of Book 1, Euclid uses a premise that was neither postulated nor proved: that two circles with centers at the distance of their radius will intersect in two points. Later, in the fourth construction, he uses superposition (moving the triangles on top of each other) to prove that if two sides and their angles are equal then they are congruent. During these considerations, he uses some properties of superposition, but these properties are not constructs explicitly in the treatise. If superposition is to be considered a valid method of geometric proof, all of geometry would be full of such proofs. For example, propositions 1.1 – 1.3 can be proved trivially by using superposition.

Mathematician and historian W. W. Rouse Ball puts these criticisms in perspective, remarking that "the fact that for two thousand years ["The Elements"] was the usual text-book on the subject raises a strong presumption that it is not unsuitable for that purpose."

0.4. Open Textbooks

[This chapter has been adapted from an entry in Wikipedia.[13]]

An open textbook is a textbook licensed under an open copyright license and made available online to be freely used by students, teachers and members of the public. Many open textbooks are distributed in other printed, e-book, or audio formats that may be downloaded or purchased at little or no cost.

Part of the broader open educational resources movement, open textbooks increasingly are seen as a solution to challenges with traditionally published textbooks, such as access and affordability concerns. Open textbooks were identified in the New Media Consortium's *2010 Horizon Report* as a component of the rapidly progressing adoption of open content in higher education.

0.4.1. Usage Rights. The defining difference between open textbooks and traditional textbooks is that the copyright permissions on open textbooks allow the public to freely use, adapt, and distribute the material. Open textbooks either reside in the public domain or are released under an open license that grants usage rights to the public so long as the author is attributed.

The copyright permissions on open textbooks extend to all members of the public and cannot be rescinded. These permissions include the right to do the following:

[13] https://en.wikipedia.org/wiki/Open_textbook

- use the textbook freely
- create and distribute copies of the textbook
- adapt the textbook by revising it or combining it with other materials

Some open licenses limit these rights to non-commercial use or require that adapted versions be licensed the same as the original.

0.4.2. Open Licenses. Some examples of open licenses are:

- Creative Commons Attribution (CC-BY)
- Creative Commons Attribution Share-Alike (CC-BY-SA)
- Creative Commons Attribution Non-Commercial Share-Alike (CC-BY-NC-SA)
- GNU Free Documentation License

Waivers of copyright that place materials in the public domain include:

- Creative Commons Public Domain Certification

0.4.3. Affordability. Open textbooks increasingly are seen as an affordable alternative to traditional textbooks in both K-12 and higher education. In both cases, open textbooks offer both dramatic up-front savings and the potential to drive down traditional textbook prices through competition.

0.4.3.1. *Higher Education.* Overall, open textbooks have been found by the Student PIRGs to offer 80% or more savings to higher education students over traditional textbook publishers. Research commissioned by the Florida State Legislature found similarly high savings and the state has since implemented a system to facilitate adoption of open textbooks.

In the Florida legislative report, the governmental panel found after substantial consultation with educators, students, and administrators that "there are compelling academic reasons to use open access textbooks such as: improved quality, flexibility and access to resources, interactive and active learning experiences, currency of textbook information, broader professional collaboration, and the use of teaching and learning technology to enhance educational experiences." (OATTF, p. i) Similar state-backed initiatives are underway in Washington, Ohio, California, and Texas. In Canada, the province of British Columbia became the first jurisdiction to have a similar open textbook program.

0.4.3.2. *K-12 Education.* Research at Brigham Young University has produced a web-based cost comparison calculator for traditional and open K-12 textbooks. To use the calculator, input commercial textbook cost, planned replacement frequency, and number of annual textbook user count are required. A section is provided to input time requirements for adaptation to local needs, annual updating hours, labor rate, and an approximation of pages. The summary section applies an industry standard cost for print-on-demand of the adapted open textbook to provide a cost per student per year for both textbook options. A summed cost differential over the planned period of use is also calculated.

0.4.4. Milestones. In November 2010, Dr. Anthony Brandt was awarded an "Access to Artistic Excellence" grant from the National Endowment for the Arts for his innovative music appreciation course in Connexions. "Sound Reasoning ... takes a new approach [to teaching music appreciation]: It presents style-transcendent principles, illustrated by side-by-side examples from both traditional and contemporary music. The goal is to empower listeners to be able to listen attentively and think intelligently about any kind of music, no matter its style. Everything is listening based; no ability to read music is required." The module being completed with grant funds is entitled "Hearing Harmony". Dr. Brandt cites choosing the Connexions open content publishing platform because "it was an opportunity to present an innovative approach in an innovative format, with the musical examples interpolated directly into the text."

In December 2010, open textbook publisher Flat World Knowledge was recognized by the American Library Association's Business Reference and Services Section (ALA BRASS) by being named to the association's list of "Outstanding Business Reference Sources: The 2010 Selection of Recent Titles." The categories of business and economics open textbooks from Flat World Knowledge's catalog were selected for this award and referenced as "an innovative new vehicle for affordable (or free) online access to premier instructional resources in business and economics." Specific criteria used by the American Library Association BRASS when evaluating titles for selection were:

> A resource compiled specifically to supply information on a certain subject or group of subjects in a form that will facilitate its ease of use. The works are examined for authority and reputation of the publisher, author, or editor; accuracy; appropriate bibliography; organization, comprehensiveness,

> and value of the content; currency and unique addition to the field; ease of use for intended purpose; quality and accuracy of indexing; and quality and usefulness of graphics and illustrations. Each year more electronic reference titles are published, and additional criteria by which these resources are evaluated include search features, stability of content, graphic design quality, and accuracy of links. Works selected are intended to be suitable for medium to large academic and public libraries.

Because authors do not make money form the sale of open textbooks, several organizations have tried to use prizes or grants as financial incentives for writing open textbooks or releasing existing textbooks under open licenses. Connexions announced a series of two grants in early 2011 that will allow them to produce a total of 20 open textbooks. The first five titles will be produced over an 18 month time frame for Anatomy & Physiology, Sociology, Biology, Biology for non-majors, and Physics. The second phase will produce an additional 15 titles with subjects that have yet to be determined. It is noted the most expensive part of producing an open textbook is image rights clearing. As images are cleared for this project, they will be available for reuse in even more titles. In addition, the Saylor Foundation sponsored an "Open Textbook Challenge", offering a $20,000 reward for newly-written open textbooks or existing textbooks released under a CC-BY license.

The Text and Academic Author's Association awarded a 2011 Textbook Excellence Award ("Texty") to the first open textbook to ever win such recognition this year. A maximum of eight academic titles can earn this award each year. The title "Organizational Behavior" by Talya Bauer and Berrin Erdogan earned one of seven 2011 Textbook Excellence Awards granted. Bauer & Erdogan's "Organizational Behavior" open textbook is published by Flat World Knowledge.

0.4.5. Instruction. Open textbooks are flexible in ways that traditional textbooks are not, which gives instructors more freedom to use them in the way that best meets their instructional needs.

One common frustration with traditional textbooks is the frequency of new editions, which force the instructor to modify the curriculum to the new book. Any open textbook can be used indefinitely, so instructors need only change editions when they think it is necessary.

Many open textbooks are licensed to allow modification. This means that instructors can add, remove or alter the content to better fit a course's needs. Furthermore, the cost of textbooks can in some cases contribute to the quality of instruction when students are not able to purchase required materials. A Florida governmental panel found after substantial consultation with educators, students, and administrators that "there are compelling academic reasons to use open access textbooks such as: improved quality, flexibility and access to resources, interactive and active learning experiences, currency of textbook information, broader professional collaboration, and the use of teaching and learning technology to enhance educational experiences." (OATTF, p. i)

0.4.6. Authorship. Author compensation for open textbooks works differently than traditional textbook publishing. By definition, the author of an open textbook grants the public the right to use the textbook for free, so charging for access is no longer possible. However, numerous models for supporting authors are developing. For example, a start-up open textbook publisher called Flat World Knowledge pays its authors royalties on the sale of print copies and study aids. Other proposed models include grants, institutional support and advertising.

0.4.7. Projects. A number of projects seek to develop, support and promote open textbooks. Two very notable advocates and supporters of open textbook and related open education projects include the William and Flora Hewlett Foundation and the Bill and Melinda Gates Foundation.

0.5. Recommended Reading

Book of Proof, 2nd edition, Richard Hammack.

This open textbook provides an introduction to the standard methods of proving mathematical theorems. It can be considered a companion volume to any edition of Euclid, especially for those who are learning how to read and write mathematical proofs for the first time. It has been approved by the American Institute of Mathematics' Open Textbook Initiative and has a number of good reviews at the Mathematical Association of America Math DL and on Amazon. Visit the website at:

`http://www.people.vcu.edu/~rhammack/BookOfProof/index.html`

`http://www.amazon.com/Book-Proof-Richard-Hammack/dp/0989472108/`

Euclid's *Elements* online, coded and maintained by David E. Joyce

http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

Euclid's Elements – All thirteen books in one volume

Based on Heath's translation, Green Lion Press, ISBN 978-1888009194.

Geometry Without Multiplication, Victor Aguilar

This geometry textbook will be the first in a series (the second being Geometry With Multiplication). It will not only be suitable for high school students but will also maintain the rigor required in college level textbooks. The most recent draft can be found at:

https://www.researchgate.net/publication/291333791_Volume_One_Geometry_without_Multiplication

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Guidelines for Good Mathematical Writing, Francis Edward Su

Download at:

https://www.math.hmc.edu/~su/math131/good-math-writing.pdf

How to Solve it, G. Polya

"A perennial bestseller by eminent mathematician G. Polya, *How to Solve It* will show anyone in any field how to think straight. In lucid and appealing prose, Polya reveals how the mathematical method of demonstrating a proof or finding an unknown can be of help in attacking any problem that can be 'reasoned' out – from building a bridge to winning a game of anagrams. Generations of readers have relished Polya's deft – indeed, brilliant – instructions on stripping away irrelevancies and going straight to the heart of the problem."[14]

http://www.amazon.com/How-Solve-Mathematical-Princeton-Science/dp/069116407X/

Khan Academy

https://www.khanacademy.org/

Math Open Reference, especially the topic of Triangle Centers

[14]This description (c)1985 by Princeton University Press.

http://www.mathopenref.com/

http://www.mathopenref.com/trianglecenters.html

Oliver Byrne: The First Six Books of the Elements of Euclid

Taschen Publications, ISBN-13: 978-3836544719

A beautiful, full-color edition of Euclid.

The Thirteen Books of Euclid's Elements, translation and commentaries by Sir Thomas Heath in three volumes.

Published by Dover Publications, Vol. 1: ISBN 978-0486600888, Vol. 2: ISBN 978-0486600895, Vol. 3: ISBN 978-0486600901.

CHAPTER 1

Angles, Parallel Lines, Parallelograms

Students are expected to construct figures and/or work through the proofs step-by-step. This is an essential component to the learning process that cannot be avoided. In fact, a king once asked Euclid for a faster way to learn mathematics than the Elements, and Euclid replied, "There is no royal road to geometry", which means: no one learns mathematics for free.

1.1. Symbols, Logic, and Definitions

The propositions of Euclid will be referred to in brackets; for example, we[1] will usually write [3.32] instead of writing Proposition 3.32. Axioms, Definitions, etc., will also be referred to in this way; for example, Definition 12 in chapter 1 will be written as [Def. 1.12]. Exercises to problems will be written as [3.5, #1] instead of exercise 1 of Proposition 3.5.

Numbered equations will be written as (10.2.2) instead of the second equation in chapter 10, section 2.

A note on the exercises: you don't have to do all or even most of them. Generally, an exercise is expected to be solved using the propositions, corollaries, and exercises that preceded it. For example, exercise [1.32, #3] should first be attempted using propositions [1.1]-[1.32] as well as all previous exercises. Should this prove too difficult or too frustrating for the student, then he/she should consider whether propositions [1.33] or later (and their exercises) might help solve the exercise. It is also permissible to use trigonometry, linear algebra, or other contemporary mathematical techniques on challenging problems.

1.1.1. Symbols. The following symbols will be used to denote standard geometric shapes or relationships:

[1]Mathematicians tend to write "we" when writing about math in the same way that coaches will tell players "here's what we're going to do". Obviously, the players will be doing something while the coaches watch, but the coach says "we" to help everyone remember they are engaged in a team effort. A mathematician will say or write "we" because everyone who is attempting to understand the problem is on a kind of team where we all work and struggle together.

- Circles will usually be written as: $\bigcirc$. When the center of a circle is known (for example, point A), the circle will be identified as $\bigcirc A$. Otherwise, the circle will be identified with points on its circumference, such as $\bigcirc ABC$.
- Triangles by: $\triangle$
- Parallelograms by: $\boxdot$
- Parallel lines by: $\parallel$
- Perpendicular lines by: $\perp$

In addition to these, we shall employ the usual symbols of algebra: $+$, $-$, $=$, $<$, $>$, $\leq$, $\geq$, $\neq$, as well as a few additional symbols:

- Composition: $\oplus$ For example, suppose we have the segments $\overline{AB}$ and $\overline{BC}$ which intersect at the point B. The statement $\overline{AB}+\overline{BC}$ refers to the sum of their lengths, but $\overline{AB} \oplus \overline{BC}$ refers to their composition as one object. See Fig. 1.1.1.

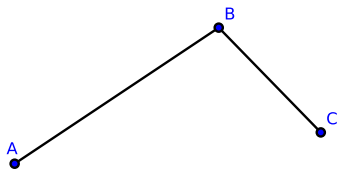

FIGURE 1.1.1. Composition: the geometrical object $\overline{AB} \oplus \overline{BC}$ is a single object composed of two segments, $\overline{AB}$ and $\overline{BC}$.

The composition of angles, however, can be written using either $+$ or $\oplus$, and in this textbook their composition will be written with $+$.

- Similar: $\sim$ Two figures or objects are similar if they have the same shape but not necessarily the same size. If two similar objects have the same size, they are also congruent.
- Congruence: $\cong$ Two figures or objects are congruent if they have the same shape and size, or if one has the same shape and size as the mirror image of the other. This means that an object can be repositioned and reflected so as to coincide precisely with the other object (but it doesn't have to be re-sized).[2]

[2] http://en.wikipedia.org/wiki/Congruence_(geometry)

1.1.2. Logic. *Propositions* are mathematical statements which may be divided into theorems and axioms.

An *axiom* is a proposition regarded as self-evidently true without proof[3]. Axioms are incapable of demonstration; they are considered so fundamental that they cannot be inferred from any proposition which is more elementary. "Any two sides of a triangle are greater in length than the third side" may be self-evident; however, it is not an axiom since it can be inferred by demonstration from other propositions. The statement "two objects which are equal in length to a third object are also equal in length to each other" is self-evident, and so it is considered an axiom. (Of course, whether a given proposition is an axiom or a theorem depends on which textbook you are reading. Graduate textbooks are famous for asking students to prove propositions which were considered axioms at the undergraduate level.)

A *theorem* is the formal statement of a property that may be demonstrated from known propositions (these propositions may themselves be theorems or axioms). A theorem consists of two parts: the *hypothesis*, or that which is assumed, and the *claim*, or that which the author will prove using a logical argument. Four examples follow:

THEOREM. *(1) If X is Y, then Z is W.*

The hypothesis is that X is Y, and the claim is that Z is W. The author begins by assuming that X is Y; the bulk of the proof will show logically that because X is Y we must also have that Z is W.

Converse Theorems: Two theorems are said to be converses when the hypothesis and claim of the first theorem become the claim and the hypothesis of the second theorem. Thus[4] the converse of the theorem (1) is:

THEOREM. *(2) If Z is W, then X is Y.*

From two theorems (1) and (2), we may infer two others called their *contrapositives*. The contrapositive of (1) is:

THEOREM. *(3) If Z is not W, then X is not Y.*

The contrapositive of (2) is:

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[3]Source: Weisstein, Eric W. "Axiom." From MathWorld Wolfram Web Resource. `http://mathworld.wolfram.com/Axiom.html`

[4]"Thus" is one of three words that mathematicians use to mean "consequently" or "for this reason"; these three words are *hence*, *thus*, and *therefore*. Generally, "hence" is used for small results, "therefore" is used for big results, and "thus" is used for the results in between.

THEOREM. *(4) If X is not Y, then Z is not W.*

Theorem (4) is also called the inverse[5] of (1), and (3) is the inverse of (2).

A *corollary* is an inference or deduction based a proposition which usually states a small but important result that follows immediately from the proof itself or from the result of the proposition. For example, if a theorem states that all odd numbers are not divisible by 2, then one corollary to this theorem is that 3 is not divisible by 2 because 3 is an odd number.

A *lemma* is a proven proposition which is used as a stepping stone to a larger result rather than as a statement of interest by itself[6]. For example, if we wish to prove the side-side-side theorem for the congruence of triangles, we may wish to first prove that any two triangle with the same base but a different third vertex do not have the same shape (see [1.7] and [1.8]).

1.1.3. Definitions. We all need a common language in order to discuss similar experiences or ideas. For mathematics, this is true to an almost ridiculous degree. A mathematician's work can be rendered useless if the definitions he or she begins with turn out to be vague or sloppy.

We now state the definitions of a number of ideas that we will use regularly throughout the book.

The Point. 1. A *point* is a zero dimensional object.[7] A geometrical object which has three dimensions (length, height, and width) is a solid. A geometrical object which has two dimensions (length and height) is a surface, and a geometrical object which has one dimension is a line or line segment. Since a point has none of these, it has zero dimensions.

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[5]"The counterpart of a proposition obtained by exchanging the affirmative for the negative quality of the whole proposition and then negating the predicate: The inverse of "'Every act is predictable' is 'No act is unpredictable.'"
The American Heritage® Dictionary of the English Language, Fourth Edition copyright ©2000 by Houghton Mifflin Company. Updated in 2009. Published by Houghton Mifflin Company. All rights reserved.

[6]Source: https://en.wikipedia.org/wiki/Lemma_(mathematics)

[7]Warren Buck, Chi Woo, Giangiacomo Gerla, J. Pahikkala. "point" (version 13). PlanetMath.org. Freely available at http://planetmath.org/point

The Line. 2. A *line* is a one dimensional object: it has only length. If it had any height or width, no matter how small, it would have two dimensions. Hence, a line has neither height nor width.

A line with points A and B is written as $\overleftrightarrow{AB}$.

(This definition conforms to Euclid's original definition in which a line need not be straight. However, in all modern geometry texts, it is understood that a "line" has no curves. See also [Def 1.4].)

3. The intersections of lines are points. However, a point may exist without being the intersection of lines.

4. A line without a curve is called a *straight line*. It is understood throughout this textbook that a *line* refers exclusively to a *straight line*. A curved line (such as the circumference of a circle) will never be referred to merely as a line in order to avoid confusion. Lines have no endpoints since they are infinite in length.

A *line segment* (or more simply a *segment*) is similar to a line except that it is finite in length and has two *endpoints* at its extremities. A line segment with endpoints A and B is written as $\overline{AB}$.

A *ray* is like a line in that it is infinite in length; however, it has one endpoint. A ray with endpoint A and point B is written as $\overrightarrow{AB}$ (where A is the endpoint).

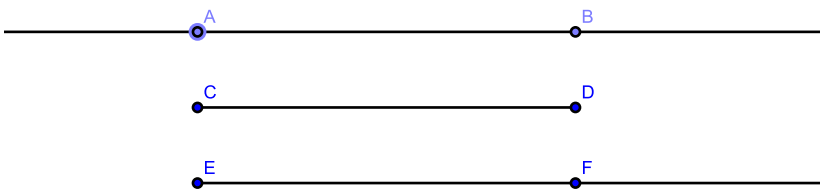


FIGURE 1.1.2. [Def. 1.2, 1.3, 1.4] $\overleftrightarrow{AB}$ is a line, $\overline{CD}$ is a line segment, and $\overrightarrow{EF}$ is a ray

The Plane. 5. A *surface* has two dimensions: length and height. It has no width; if it had, it would be a space of three dimensions.

6. A *plane* is a surface that extends infinitely far and is assumed to be completely flat. A plane is the two-dimensional analogue of a point (zero dimensions), a line (one dimension) and three-dimensional space. Planes act as the

setting of Euclidean geometry; that is, "the plane" refers to the whole space in which two-dimensional geometry is performed.

Planes are defined by three points. For any three points not on the same line, there exists one and only one plane which contains all three points.

7. Any combination of points, lines, line segments, or curves on a plane is called a *plane figure*. A plane figure that is bounded by a finite number of straight line segments closed in a loop to form a closed chain or circuit is called a *polygon*[8].

All bounded plane figures have a measure called *area*. Area[9] is the quantity that expresses the extent of a two-dimensional figure or shape on a plane. Area can be understood as the amount of material with a given thickness that would be necessary to fashion a model of the shape, or the amount of paint necessary to cover the surface with a single coat.

Area is the two-dimensional analog of the length of a curve (a one-dimensional concept) or the volume of a solid (a three-dimensional concept). Surface area is its analog on the two-dimensional surface of a three-dimensional object.[10]

8. Points which lie on the same straight line, ray, or segment are called *collinear points*.

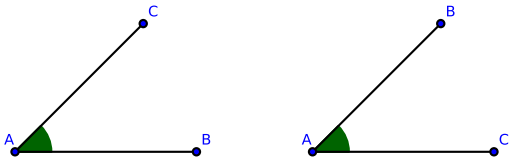


FIGURE 1.1.3. [Def 1.11] Notice that both angles could be referred to as $\angle BAC$, $\angle CAB$, or the angle at point A where A is a vertex.

The Angle. 9. The angle made by of two straight lines, segments, or rays extending outward from a common point but in different directions is called a *rectilinear angle* (or simply an *angle*).

10. The one point of intersection between straight lines, rays, or segments is called the *vertex of the angle*.

11. A particular angle in a figure will be written as the symbol $\angle$ and three letters, such as BAC, of which the middle letter, A, is at the vertex. Hence,

[8] http://en.wikipedia.org/wiki/Polygon
[9] Taken from the article: https://en.wikipedia.org/wiki/Area.
[10] See the List of Formulas in: https://en.wikipedia.org/wiki/Area

such an angle may be referred to either as $\angle BAC$ or $\angle CAB$. Occasionally, this notation will be shortened to "the angle at point A" instead of naming the angle as above.

12. The angle formed by composing two or more angles is called their sum. Thus in Fig. 1.1.4, we have that $\angle ABC \oplus \angle PQR = \angle ABR$ where the segment $\overline{QP}$ is applied to the segment $\overline{BC}$. We generally choose to write $\angle ABC + \angle PQR = \angle ABR$ to express this concept.

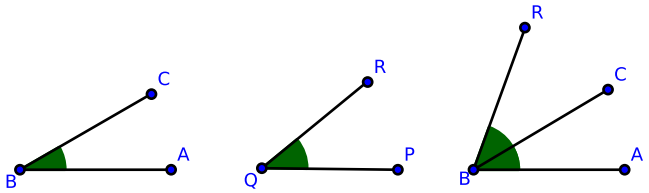


FIGURE 1.1.4. [Def. 1.12]

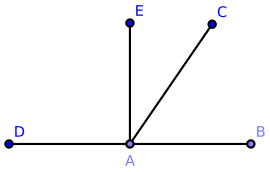


FIGURE 1.1.5. [Def. 1.13]

13. Suppose two segments $\overline{BA}$, $\overline{AD}$ are composed such that $\overline{BA} \oplus \overline{AD} = \overline{BD}$ where $\overline{BD}$ is a segment. If a point C which is not on the segment $\overline{BD}$ is connected to point A, then the angles $\angle BAC$ and $\angle CAD$ are called *supplements* of each other (see Fig. 1.1.5). This definition holds when we replace segments by straight lines or rays, *mutatis mutandis*[11].

14. When one segment, $\overline{AE}$, stands on another segment, $\overline{DB}$, such that the adjacent angles on either side of $\overline{AE}$ are equal (that is, $\angle EAD = \angle EAB$), each of the angles is called a *right angle*, and the segment which stands on the other is described as *perpendicular* to the other (or sometimes *the perpendicular* to the other). See Fig. 1.1.5.

We may write that $\overline{AE}$ is perpendicular to $\overline{DB}$ or more simply that $\overline{AE} \perp \overline{DB}$. It follows that the supplementary angle of a right angle is another right angle.

[11] *Mutatis mutandis* is a Latin phrase meaning "changing [only] those things which need to be changed" or more simply "[only] the necessary changes having been made". Source: `http://en.wikipedia.org/wiki/Mutatis_mutandis`

Multiple perpendicular lines on a many-sided object may be referred to as the object's *perpendiculars*.

The above definition holds for straight lines and rays, *mutatis mutandis*.

A line segment within a triangle that runs from a vertex to an opposite side and is perpendicular to that side is usually referred to an *altitude* of the triangle, although it could be referred to in a general sense as a perpendicular of the triangle.

15. An *acute angle* is one which is less than a right angle. $\angle DAB$ in Fig. 1.1.6 is an acute angle.

16. An *obtuse angle* is one which is greater than a right angle. $\angle EAB$ in Fig. 1.1.6 is an obtuse angle. The supplement of an acute angle is obtuse, and conversely, the supplement of an obtuse angle is acute.

17. When the sum of two angles is a right angle, each is called the *complement* of the other. See Fig. 1.1.6.

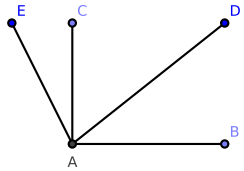


FIGURE 1.1.6. [Def. 1.17] The angle $\angle BAC$ is a right angle. Since $\angle BAC = \angle CAD + \angle DAB$, it follows that the angles $\angle BAD$ and $\angle DAC$ are each complements of the other.

Concurrent Lines. 18. Three or more straight lines intersecting the same point are called *concurrent* lines. This definition holds for rays and segments, *mutatis mutandis*.

19. The common point through which the rays pass is called the vertex.

The Triangle. 20. A triangle is a polygon formed by three segments joined at their endpoints. These three segments are called the *sides* of the triangle. One side in particular may be referred to as the *base* of the triangle for explanatory reasons, but usually there is no fundamental difference between the properties of a base and the properties of either of the two remaining sides of a triangle. An exception to this is the isosceles triangle where two sides are equal in length: the third side which is not equal is denoted as the base.

The formula for the area of a triangle is defined as

$$A = \frac{1}{2}bh$$

where b =the length of any side, and h =the length of a perpendicular segment from the vertex opposite the side onto the line which contains the side in question.

The area of a triangle is zero if and only if its vertices are collinear.

21. A triangle whose three sides are unequal in length is called *scalene* (the left-hand triangle in Fig. 1.1.7). A triangle with two equal sides is called *isosceles* (the middle triangle in Fig. 1.1.7). When all sides are equal, a triangle is called *equilateral*, (the right-hand triangle in Fig. 1.1.7). When all angles are equal, a triangle is called *equiangular*.

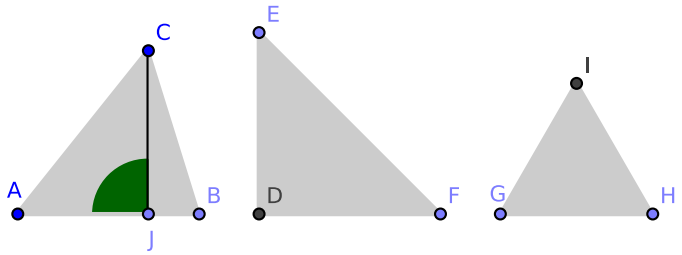


FIGURE 1.1.7. [Def 1.21] The three types of triangles: *scalene, isosceles, equilateral*.

22. A *right triangle* is a triangle in which one of its angles is a right angle, such as the middle triangle in Fig. 1.1.7. The side which stands opposite the right angle is called the *hypotenuse* of the triangle. (In the middle triangle in Fig. 1.1.7, $\angle EDF$ is a right angle, so side EF is the hypotenuse of the triangle.)

Notice that side EF of a triangle is not written as $\overline{EF}$ despite the fact that EF is also a line segment; since EF is a side of a triangle, we may omit the overline.

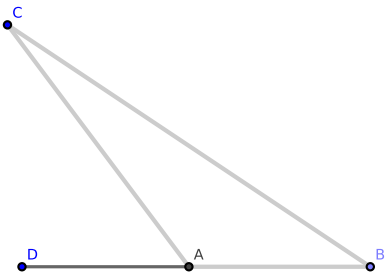


FIGURE 1.1.8. [Def. 1.23]

23. An *obtuse* triangle is a triangle such that one of its angles is obtuse (such as $\angle CAB$ in $\triangle CAB$, Fig. 1.1.8).

24. An *acute triangle* is a triangle such that each of its angles are acute, such as the left and right triangles in Fig. 1.1.7.

25. An *exterior angle* of a triangle is one which is formed by extending the side of a triangle. For example, the triangle in Fig. 1.1.8 has had side BA extended to the segment $\overline{BD}$ which creates the exterior angle $\angle DAC$.

Every triangle has six exterior angles. Also, each exterior angle is the supplement of the adjacent interior angle. In Fig. 1.1.8, the exterior angle $\angle DAC$ is the supplement of the adjacent interior angle $\angle CAB$.

The Polygon. 26. A *rectilinear figure* bounded by three or more line segments can also be referred to as a polygon (see definition 7). For example, a circle is a plane figure but not a polygon, but the triangles in Fig. 1.1.8 are both plane figures and polygons.

27. A polygon is said to be *convex* when it does not have an interior angle greater than $180°$.

28. A polygon of four sides is called a *quadrilateral*.

29. A *lozenge* is an equilateral parallelogram whose acute angles are 45 degrees. Sometimes, the restriction to 45 degrees is dropped, and it is required only that two opposite angles are acute and the other two obtuse. The term *rhombus* is commonly used for an equilateral parallelogram; see Fig. 1.1.9.[12]

[12]Source: Weisstein, Eric W. "Lozenge." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/Lozenge.html`

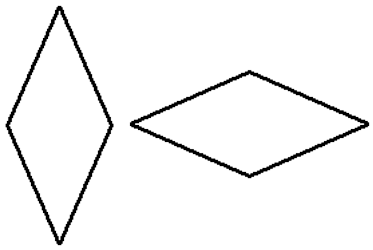


FIGURE 1.1.9. [Def. 1.29] Two rhombi.

30. A rhombus which has a right angle is called a *square*.

31. A polygon which has five sides is called a *pentagon*; one which has six sides, a *hexagon*, etc.[13]

The Circle. 32. A *circle* is a plane figure formed by a curved line called the *circumference* such that all segments constructed from a certain point within the figure to the circumference are equal in length. That point is called the *center* of the circle.

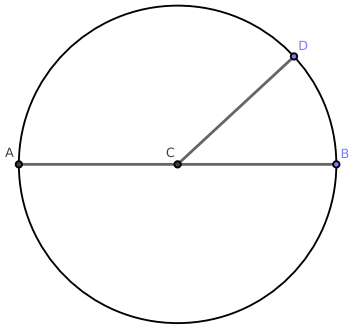


FIGURE 1.1.10. [Def. 1.32] $\bigcirc C$ is constructed with center C and radius $\overline{CD}$. Notice that $\overline{CA} = \overline{CB} = \overline{CD}$. Also notice that $\overline{AB}$ is a diameter.

33. A *radius* of a circle is any segment constructed from the center to the circumference, such as $\overline{CA}$, $\overline{CB}$, $\overline{CD}$ in Fig. 1.1.10. Notice that $\overline{CA} = \overline{CB} = \overline{CD}$.

34. A *diameter* of a circle is a segment constructed through the center and terminated at both ends by the circumference, such as $\overline{AB}$ in Fig. 1.1.10.

[13]See also `https://en.wikipedia.org/wiki/Polygon`

From the definition of a circle, it follows that the path of a movable point in a plane which remains at a constant distance from a fixed point is a circle. Also, any point P in the plane is either inside, outside, or on the circumference of a circle depending on whether its distance from the center is less than, greater than, or equal to the radius.

Other. 35. A segment, line, or ray in any figure which divides the area of a regular or symmetrical geometric object into two equal halves is called an *Axis of Symmetry* of the figure (such as AC in the polygon $ABCD$, Fig. 1.1.11).

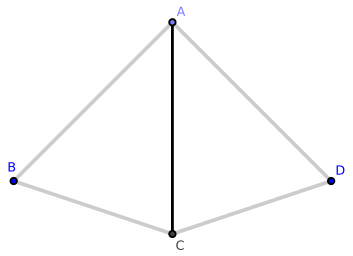


FIGURE 1.1.11. [Def. 1.35]

Alternatively, if an object is bisected by a segment (or line or ray) such that if for each point on one side of the segment (or line or ray) there exists one point on the other side of the segment (or line or ray) where the distance from each of these points to the segment (or line or ray) is equal, then the segment (or line or ray) is an *Axis of Symmetry.*

36. A segment constructed from any angle of a triangle to the midpoint of the opposite side is called a *median of the triangle*. Each triangle has three medians which are concurrent. The point of intersection of the three medians is called the *centroid* of the triangle.

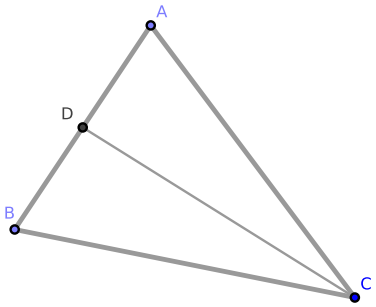


FIGURE 1.1.12. [Def. 1.36] $\overline{CD}$ is a median of $\triangle ABC$. The triangle has two other medians not shown, and their intersection is the centroid of $\triangle ABC$.

37. A *locus* (plural: loci) is a set of points whose location satisfies or is determined by one or more specified conditions, i.e., 1) every point satisfies a given condition, and 2) every point satisfying it is in that particular locus.[14] For example, a circle is the locus of a point whose distance from the center is equal to its radius.

1.2. Postulates

We assume the following:

(1) A straight line, ray, or segment may be constructed from any one point to any other point. Lines, rays, and segments may be subdivided by points into segments or subsegments which are finite in length.
(2) A segment may be extended from any length to a longer segment, a ray, or a straight line.
(3) A circle may be constructed from any point (which we denote as its center) and from any finite length measured from the center (which we denote as its radius).

Note: if we have constructed two points A and B on a sheet of paper, and if we construct a segment from A to B, this segment will have some irregularities due to the spread of ink or slight flaws in the paper, both of which introduce some height and width. Hence, it will not be a true geometrical segment no matter how nearly it may appear to be so. This is the reason that Euclid postulates the construction of segments, rays, and straight lines from one point to another (where our choice of paper, application, etc., is irrelevant). For if a

[14] http://en.wikipedia.org/wiki/Locus_(mathematics)

segment could be accurately constructed, there would be no need for Euclid to ask us to take such an action for granted. Similar observations apply to the other postulates. It is also worth nothing that Euclid never takes for granted the accomplishment of any task for which a geometrical construction, founded on other problems or on the foregoing postulates, can be provided.

1.3. Axioms

1.3.1. Algebraic Axioms. Let a, b, c, etc., be real numbers. These axioms require four operations: addition, subtraction, multiplication, and subtraction.

Addition[15] (often signified by the plus symbol "+") is one of the four basic operations of arithmetic, with the others being subtraction, multiplication and division. The addition of two numbers which represent quantities gives us the total amount of those quantities combined. For example, if $\angle A = \frac{3}{4}\pi$ radians and $\angle B = \frac{1}{4}\pi$ radians, then

$$\angle A + \angle B = \frac{3}{4}\pi + \frac{1}{4}\pi = \left(\frac{3}{4} + \frac{1}{4}\right)\pi = \frac{4}{4}\pi = \pi$$

radians.

Addition for quantities is defined when the quantities under consideration have the same units. For example, a segment with a length of 3 units that is extended by an additional 7 units now has side-length of 10 units; or $3+7=10$ where each number represents the number of units. However, if a segment with a length of 3 is added to $\angle A = \frac{3}{4}\pi$ radians, the sum is not defined. (In some cases, if the units are removed and the now unit-less numbers are added together, their sum is defined, but the sum may have no bearing on the problem the geometer wishes to solve.)

Subtraction represents the operation of removing quantities from a collection of quantities. It is signified by the minus sign ($-$). For example, if $\angle A = \frac{3}{4}\pi$ radians and $\angle B = \frac{1}{4}\pi$ radians, then

$$\angle A - \angle B = \frac{3}{4}\pi - \frac{1}{4}\pi = \left(\frac{3}{4} - \frac{1}{4}\right)\pi = \frac{2}{4}\pi = \frac{1}{2}\pi$$

radians.

Subtraction is a special case of addition where

$$a + (-b) = a - b = c$$

[15]Some of this and the following text comes from the relevant articles in Wikipedia.

Hence, the caveat above concerning units applies to subtraction.

Multiplication (often denoted by the cross symbol "$\times$", by a point "$\cdot$" or by the absence of symbol): when thinking of multiplication as repeated addition, multiplication is equivalent to adding as many copies of one of them (multiplicand) as the value of the other one (multiplier). Normally the multiplier is written first and multiplicand second, though this can vary and sometimes the distinction is not meaningful. As one example,

$$a \times b = a + a + \ldots + a$$

where the product $a \times b$ equals a added to itself a total of b times.

Multiplication is a special case of addition, and so the caveat above concerning units applies to multiplication. The area of a certain geometric objects on the plane (triangles, rectangles, parallelograms, etc.) can be defined by the product of two lengths (base and height, two adjacent side-lengths, etc.). The volume of a certain solid geometric objects (spheres, cubes, etc.) can also be defined by the product of certain lengths.

Division: in elementary arithmetic, division (denoted $\div$ or $/$ or by $\frac{a}{b}$ where $b \neq 0$) is an arithmetic operation. Specifically, if b times c equals a, written:

$$a = b \times c$$

where b is not zero, then a divided by b equals c, written:

$$a \div b = c \qquad a/b = c \qquad \frac{a}{b} = c$$

Division is a special case of multiplication where

$$a \cdot \frac{1}{b} = a \div b = \frac{a}{b}$$

Hence, the caveat above concerning units applies to division.

We use the following as algebraic axioms:[16]

(1) The Addition Property: If $a = b$ and $c = d$, then $a + c = b + d$.
(2) The Subtraction Property: If $a = b$ and $c = d$, then $a - c = b - d$.
(3) Multiplication Property: If $a = b$, then $ca = cb$.

[16]Sources for these axioms include:
(A) Jurgensen, Brown, Jurgensen. "Geometry." Houghton Mifflin Company, Boston, 1985. ISBN: 0-395-35218-5
(B) Relevant articles in Wikipedia.

(4) Division Property: If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.
(5) Substitution Property: If $a = b$, then either a or b may be substituted for the other in any equation or inequality.
(6) Reflexive Property: $a = a$.
(7) Symmetric Property: If $a = b$, then $b = a$.
(8) Converse Properties of Inequalities:
    (a) If $a \leq b$, then $b \geq a$.
    (b) If $a \geq b$, then $b \leq a$.
(9) Transitive Properties of Inequalities:
    (a) If $a \geq b$ and $b \geq c$, then $a \geq c$.
    (b) If $a \leq b$ and $b \leq c$, then $a \leq c$.
    (c) If $a \geq b$ and $b > c$, then $a > c$.
    (d) If $a = b$ and $b > c$, then $a > c$.
(10) Inequality Properties of Addition and Subtraction:
    (a) If $a \leq b$, then $a + c \leq b + c$ and $a - c \leq b - c$.
    (b) If $a \geq b$, then $a + c \geq b + c$ and $a - c \geq b - c$.
(11) Inequality Properties of Multiplication and Division:
    (a) If $a \geq b$ and $c > 0$, then $ac \geq bc$ and $\frac{a}{c} \geq \frac{b}{c}$.
    (b) If $a \leq b$ and $c > 0$, then $ac \leq bc$ and $\frac{a}{c} \leq \frac{b}{c}$.
    (c) If $a \geq b$ and $c < 0$, then $ac \leq bc$ and $\frac{a}{c} \leq \frac{b}{c}$.
    (d) If $a \leq b$ and $c < 0$, then $ac \geq bc$ and $\frac{a}{c} \geq \frac{b}{c}$.
(12) Inequality Property of the Additive Inverse:
    (a) If $a \leq b$, then $-a \geq -b$.
    (b) If $a \geq b$, then $-a \leq -b$.
(13) Inequality Property of the Multiplication Inverse (where a and b are either both positive or both negative):
    (a) If $a \leq b$, then $\frac{1}{a} \geq \frac{1}{b}$.
    (b) If $a \geq b$, then $\frac{1}{a} \leq \frac{1}{b}$.
(14) Inequality Property of the Multiplication Inverse (where a is positive and b is negative):
    (a) If $a \geq b$, then $\frac{1}{a} \geq \frac{1}{b}$.

1.3.2. Congruence Axioms. In geometry, two figures or objects are congruent if they have the same shape and size, or if one has the same shape and size as the mirror image of the other.[17] More formally, two objects are called congruent if and only if one can be transformed into the other using only translations, rotations, or reflections. This means that either object can be repositioned and reflected (but not resized) so as to coincide precisely with

[17]Taken from `https://en.wikipedia.org/wiki/Congruence_(geometry)`

the other object. Therefore two distinct plane figures on a piece of paper are congruent if we can cut them out and then match them up completely. Turning the paper over is permitted.

Examples include:

- Two line segments are congruent if they have the same length.
- Two angles are congruent if they have the same measure.
- Two circles are congruent if they have the same diameter.

In this sense, the fact that two plane figures are congruent implies that their corresponding characteristics are "congruent" or "equal" including not just their corresponding sides and angles, but also their corresponding diagonals, perimeters and areas.

Three congruence properties with examples:

(1) Reflexive Property: $\overline{DE} \cong \overline{DE}$ and $\angle ABC \cong \angle ABC$.
(2) Symmetric Property: If $\overline{DE} \cong \overline{FG}$, then $\overline{FG} \cong \overline{DE}$. Also, if $\angle ABC \cong \angle DEF$, then $\angle DEF \cong \angle ABC$.
(3) Transitive Property: If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$. Also, if $\angle ABC \cong \angle DEF$ and $\angle DEF \cong \angle GHI$, then $\angle ABC \cong \angle GHI$.

The Algebraic and Congruence Axioms give us the Distributive Property:

$$a(b + c) = ab + ac$$

1.3.3. Geometric Axioms.

1. **Any two objects which can be made to coincide are equal in measure.** The placing of one geometrical object on another, such as a line on a line, a triangle on a triangle, or a circle on a circle, etc., is called *superposition*. The superposition employed in geometry is only mental; that is, we conceive of one object being placed on the other. And then, if we can prove that the objects coincide, we infer by the present axiom that they are equal in all respects, including magnitude. Superposition involves the following principle which, without being explicitly stated, Euclid makes frequent use: "Any figure may be transferred from one position to another without change in size or form."

2. Two straight lines cannot enclose a space.

3. All right angles are equal to each other.

4. If two lines ($\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$) intersect a third line ($\overleftrightarrow{AC}$) such that the sum of the two interior angles ($\angle BAC + \angle ACD$) on the same side is less than the sum of

two right angles, then if these lines will meet at some finite distance. (This axiom is the converse of [1.17].) See Fig. 1.3.1.

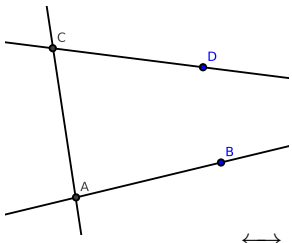


FIGURE 1.3.1. [Axiom 1.12] The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ must eventually meet (intersect) at some finite distance.

The above holds for rays and segments, *mutatis mutandis*.

Playfair's axiom[18] can also be substituted for the above axiom, which states: "In a plane, given a line and a point not on it, at most one line parallel line to the given line can be drawn through the point."

This axiom was named after the Scottish mathematician John Playfair. His "at most" clause is all that is needed since it can be proven through Euclid's propositions that at least one parallel line exists. This axiom is often written with the phrase, "there is one and only one parallel line".

Other axioms which are equivalent to axiom 4 include:

1. The sum of the angles in every triangle is 180° (triangle postulate).

2. Every triangle can be circumscribed.

3. There exists a quadrilateral in which all angles are right angles (that is, a rectangle).

4. There exists a pair of straight lines that are at constant distance from each other.

5. Two lines that are parallel to the same line are also parallel to each other.

6. There is no upper limit to the area of a triangle. (Wallis axiom)[19].

[18] https://en.wikipedia.org/wiki/Playfair's_axiom
[19] https://en.wikipedia.org/wiki/Parallel_postulate

1.4. Propositions from Book I: 1-26

PROPOSITION 1.1. *CONSTRUCTING AN EQUILATERAL TRIANGLE.*

Given an arbitrary segment, it is possible to construct an equilateral triangle on that segment.

PROOF. Suppose we are given segment $\overline{AB}$; we claim that an equilateral triangle can be constructed on $\overline{AB}$.

With A as the center of a circle and $\overline{AB}$ as its radius, we construct the circle $\bigcirc A$ [Postulate 3 from section 1.2]. With B as center and $\overline{AB}$ as radius, we construct the circle $\bigcirc B$, intersecting $\bigcirc A$ at point C.

Construct segments $\overline{CA}$, $\overline{CB}$ [Postulate 1 from section 1.2]. We claim that $\triangle ABC$ is the required equilateral triangle.

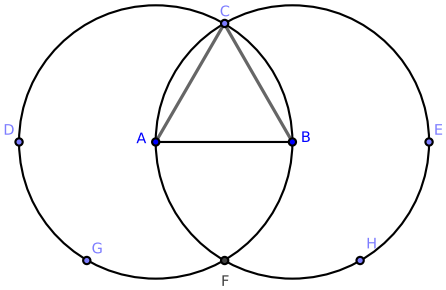


FIGURE 1.4.1. [1.1]

Because A is the center of the circle $\bigcirc A$, $\overline{AC} = \overline{AB}$ [Def. 1.33]. Since B is the center of the circle $\bigcirc B$, $\overline{AB} = \overline{BC}$. By Axiom 9 from section 1.3.1 (using equalities), we have that $\overline{AB} = \overline{AC} = \overline{BC}$.

Since these line segments are the sides of $\triangle ABC$, we have that $\triangle ABC$ is an equilateral triangle [Def. 1.21]. Since $\triangle ABC$ is constructed on segment $\overline{AB}$, we have proven our claim. □

Exam questions.

1. What do we assume in this proposition?
2. What is our claim?
3. What is a finite straight line segment?
4. What is the opposite of finite?

5. What postulates are cited and where are they cited?

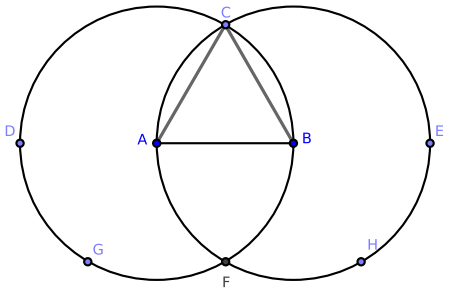
6. What axioms are cited and where are they cited?

7. What use is made of the definition of a circle? What is a circle?

8. What is an equilateral triangle?

Exercises.

The following exercises should be solved after the student has completed Chapter 1.



1. If the segments $\overline{AF}$ and $\overline{BF}$ are constructed, prove that the figure $\boxdot ACBF$ is a rhombus. [See the final chapter for a solution.]

2. If $\overline{CF}$ is constructed and $\overline{AB}$ is extended to the circumferences of the circles (at points D and E), prove that the triangles $\triangle CDF$ and $\triangle CEF$ are equilateral. [See the final chapter for a solution.]

3. If $\overline{CA}$ and $\overline{CB}$ are extended to intersect the circumferences at points G and H, prove that the points G, F, H are collinear and that the triangle $\triangle GCH$ is equilateral.

4. Construct $\overline{CF}$ and prove that $\left(\overline{CF}\right)^2 = 3 \cdot \left(\overline{AB}\right)^2$.

5. Construct a circle in the space ACB bounded by the segment $\overline{AB}$ and the partial circumferences of the two circles.

PROPOSITION 1.2. *CONSTRUCTING A LINE SEGMENT EQUAL IN LENGTH TO AN ARBITRARY LINE SEGMENT.*

Given an arbitrary point and an arbitrary segment, it is possible to construct a segment with:

(1) one endpoint being the previously given point

(2) its length is equal to that of the arbitrary segment.

PROOF. Let A be an arbitrary point on the plane, and let $\overline{BC}$ be an arbitrary segment. Our claim is stated above.

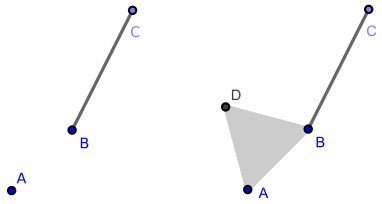


FIGURE 1.4.2. [1.2] at the beginning of the proof (left), and then partially constructed (right)

Construct $\overline{AB}$, and on $\overline{AB}$ construct the equilateral triangle $\triangle ABD$ [1.1].

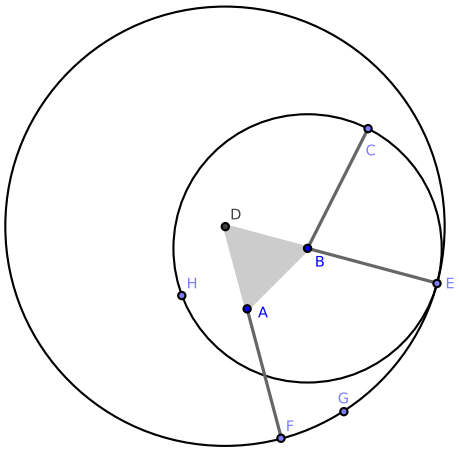


FIGURE 1.4.3. [1.2] fully constructed

With B as the center and $\overline{BC}$ as the radius, construct $\bigcirc B$. Extend $\overline{DB}$ to intersect the circle $\bigcirc B$ at E [Postulate 2 from section 1.2]. With D as the center and $\overline{DE}$ as radius, construct $\bigcirc D$. Extend $\overline{DA}$ to meet $\bigcirc B$ at F.

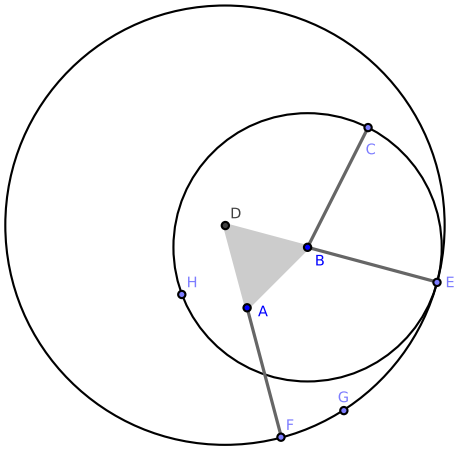
Clearly, $\overline{AF}$ has A as one of its endpoints (claim 1). If we can show that $\overline{AF} = \overline{BC}$, then we will have proven our claim.

Since $\overline{DE}$ and $\overline{DF}$ are radii of $\bigcirc D$, $\overline{DF} = \overline{DE}$ [Def. 1.32]. Because $\triangle DAB$ is an equilateral triangle, $\overline{DA} = \overline{DB}$ [Def. 1.21]. By [Axiom 2 from section 1.3.1], we have that

$$\overline{DF} - \overline{DA} = \overline{DE} - \overline{DB}$$

But $\overline{DF} - \overline{DA} = \overline{AF}$ and $\overline{DE} - \overline{DB} = \overline{BE}$. By [Axiom 5 from section 1.3.1], we have that $\overline{AF} = \overline{BE}$.

Since $\overline{BC}$ and $\overline{BE}$ are radii at $\bigcirc B$, $\overline{BC} = \overline{BE}$. By Axiom 9 from section 1.3.1 (using equalities), we have that $\overline{AF} = \overline{BC}$ (claim 2). And so we have proven our claim. □



Exercises.

1. Prove [1.2] when A is a point on $\overline{BC}$. [See the final chapter for a solution.]

PROPOSITION 1.3. *CUTTING A LINE SEGMENT AT A GIVEN SIZE.*

Given two arbitrary segments which are unequal in length, it is possible to cut the larger segment such that one of its two sub-segments is equal in length to the smaller segment.

PROOF. Let the arbitrary segments $\overline{AB}$ and $\overline{CG}$ be constructed such that $\overline{CG} < \overline{AB}$. We claim that $\overline{AB}$ may be subdivided into segments $\overline{AE}$ and $\overline{EB}$ where $\overline{AE} = \overline{CG}$.

From the point A, construct the segment $\overline{AD}$ such that $\overline{AD} = \overline{CG}$ [1.2]. With A as the center and $\overline{AD}$ as radius, construct the circle $\bigcirc A$ [Postulate 1.3] which intersects $\overline{AB}$ at E.

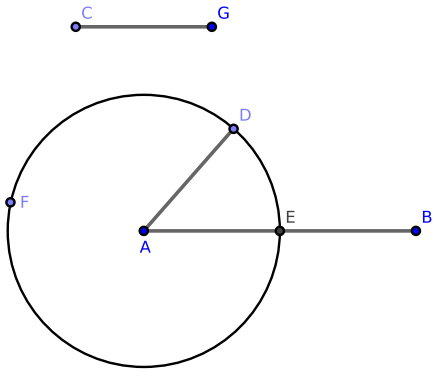


FIGURE 1.4.4. [1.3]

Because A is the center of $\bigcirc A$, $\overline{AE} = \overline{AD}$ [Def. 1.32]. Also, $\overline{AD} = \overline{CG}$ by construction. By Axiom 9 from section 1.3.1 (using equalities), we have that $\overline{AE} = \overline{CG}$ (and our claim is proved). □

COROLLARY. *1.3.1. Given arbitrary segments and a ray, it is possible to cut the ray such that the resulting segment is equal in length to the arbitrary segment.*

Exam questions.

1. What previous problem is employed in the solution of this?

2. What axiom is employed in the demonstration?

3. Demonstrate how to extend the shorter of the two given segments until the whole extended segment is equal in length to the longer segment.

Exercises.

1. Prove [Cor. 1.3.1].

PROPOSITION 1.4. *THE "SIDE-ANGLE-SIDE" THEOREM FOR THE CONGRUENCE OF TRIANGLES.*

If two pairs of sides of two triangles are equal in length, and if the triangles' corresponding interior angles are equal in measure, then the triangles are congruent.

PROOF. Suppose we have triangles $\triangle ABC$ and $\triangle DEF$ such that:

(a) the length of side AB of triangle $\triangle ABC$ is equal in length to side DE of triangle $\triangle DEF$,

(b) the length of side AC of triangle $\triangle ABC$ is equal in length to side DF of triangle $\triangle DEF$,

(c) $\angle BAC$ is the interior angle to sides AB and AC of triangle $\triangle ABC$, and $\angle EDF$ is the interior angle to sides DE and DF of triangle $\triangle DEF$,

(d) angle $\angle BAC$ in triangle $\triangle ABC$ is equal in measure to the angle $\angle EDF$ in triangle $\triangle DEF$.

We claim that $\triangle ABC \cong \triangle DEF$.

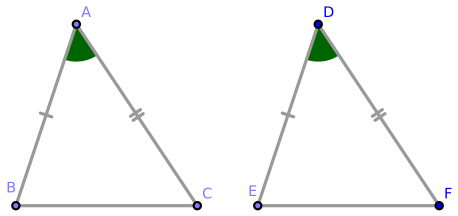


FIGURE 1.4.5. [1.4]

Recall that superposition allows us to move one object top of another without distorting its shape or measure. If $\triangle ABC$ is superpositioned[20] on $\triangle DEF$ such that the point A is positioned on the point D and side AB is positioned on side DE, then the point B coincides with the point E because $AB = DE$.

Since AB coincides with DE, the side AC also coincides with side DF because $\angle BAC = \angle EDF$. Since $AC = DF$, C coincides with F.

Because B coincides with E, the base BC of $\triangle ABC$ coincides with the base EF of $\triangle DEF$; it follows that $BC = EF$.

[20]We may also choose to write "positioned" instead of "superpositioned" with the understanding that the words are synonymous in context.

Hence all sides and angles of each triangle are equal with their corresponding sides and angles in the other triangle. We conclude that $\triangle ABC \cong \triangle DEF$. $\square$

REMARK. Euclid's Elements contains three propositions on the congruence of triangles: [1.4] side-angle-side (SAS), [1.8] side-side-side (SSS), and [1.26] angle-angle-side (AAS or SAA).

Exam questions.

1. What is meant by superposition?

2. How many parts make up a triangle? (Ans. 6, three sides and three angles.)

3. When it is required to prove that two triangles are congruent, how many parts of one must be given equal to the corresponding parts of the other? (Ans. In general, any three except the three angles. This will be established in [1.8] and [1.26], both of which use [1.4].)

Exercises.

1. Prove that the line which bisects the vertical angle of an isosceles triangle also bisects the base perpendicularly. [See the final chapter for a solution.]

2. If two adjacent sides of a quadrilateral are equal in length and the diagonal bisects the angle between them, prove that their remaining sides are also equal in length. [See the final chapter for a solution.]

3. If two segments stand perpendicularly to each other and if each bisects the other, prove that any point on either segment is equally distant from the endpoints of the other segment. [See the final chapter for a solution.]

4. If equilateral triangles are constructed on the sides of any triangle, prove that the distances between the vertices of the original triangle and the opposite vertices of the equilateral triangles are equal. (This should be proven after studying [1.32].)

PROPOSITION 1.5. *ISOSCELES TRIANGLES I.*

Suppose a triangle is isosceles. Then:

(1) if the sides of the triangle other than the base are extended, the angles under the base are equal in measure

(2) the angles at the base are equal in measure.

PROOF. Construct the triangle $\triangle ABC$ such that sides $AB = AC$ and denote side BC as the triangle's base. Extend the side AB to the segment $\overline{AD}$ and the side AC to the segment $\overline{AE}$ such that $\overline{CE} \geq \overline{BD}$. We claim that:

(1) $\angle DBC = \angle ECB$

(2) $\angle ABC = \angle ACB$

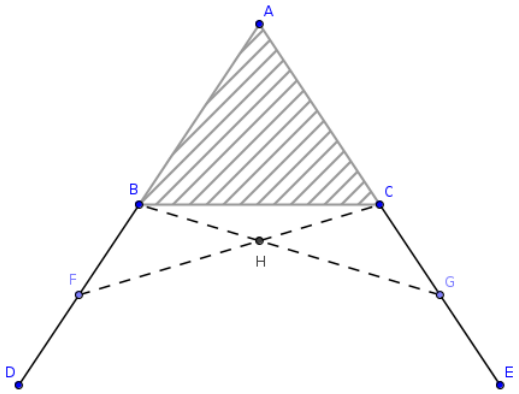


FIGURE 1.4.6. [1.5]

We will prove each claim separately. Claim 1: $\angle DBC = \angle ECB$

Let F be any point on $\overline{BD}$ other than B or D. On $\overline{CE}$, choose a point G such that $\overline{CG} = \overline{BF}$ [1.3]. (Since $\overline{CE} \geq \overline{BD}$, such a point exists.) Construct $\overline{BG}$ and $\overline{CF}$ [two applications of Postulate 1 from section 1.2].

Because $\overline{AF} = \overline{AG}$ by construction and $\overline{AB} = \overline{AC}$ by hypothesis, it follows that sides AF and AC in $\triangle FAC$ are respectively equal in length to sides AG and AB in $\triangle GAB$. Also, the angle $\angle BAC$ is the interior angle to both pairs of sides in each triangle. By [1.4], $\triangle FAC \cong \triangle GAB$; this implies that $\angle AFC = \angle AGB$ and $\overline{BG} = \overline{CF}$.

Since $\overline{BF} = \overline{CG}$, $\overline{CF} = \overline{BG}$, and $\angle AFC = \angle AGB$, by [1.4] we have that triangles $\triangle FBC \cong \triangle GCB$. This implies that $\angle FBC = \angle GCB$, which are the angles under the base of $\triangle ABC$. Or,

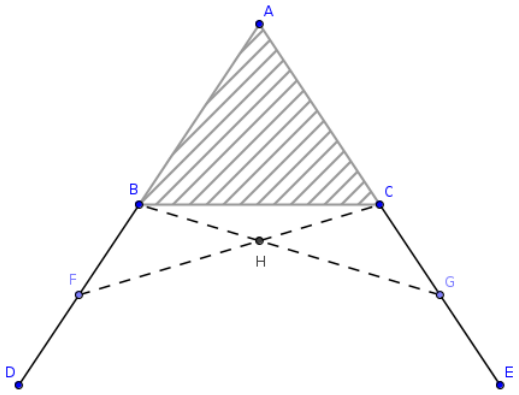
$$\angle DBC = \angle FBC = \angle GCB = \angle ECB$$

Claim 2: $\angle ABC = \angle ACB$

Since $\triangle FBC \cong \triangle GCB$, we also have that $\angle FCB = \angle GBC$. Since $\angle FCA = \angle FCB + \angle ACB$ and $\angle GBA = \angle GBC + \angle ABC$, we have that

$$\begin{aligned} \angle FCA &= \angle GBA \\ \angle FCB + \angle ACB &= \angle GBC + \angle ABC \\ \angle FCB + \angle ACB &= \angle FCB + \angle ABC \\ \angle ACB &= \angle ABC \end{aligned}$$

This proves our claim. □



Observation: The difficulty which beginners may have with this proposition is due to the fact that the two triangles $\triangle ACF$, $\triangle ABG$ overlap. A teacher or tutor should graph these triangles separately and point out the corresponding parts: $AF = AG$, $AC = AB$, and $\angle FAC = \angle GAB$. By [1.4], it follows that $\angle ACF = \angle ABG$, $\angle AFC = \angle AGB$.

COROLLARY. *1.5.1. Every equilateral triangle is equiangular.*

Exercises.

1. Prove that the angles at the base are equal without extending the sides.

2. Prove that $\overleftrightarrow{AH}$ is an Axis of Symmetry of $\triangle ABC$. [See the final chapter for a solution.]

3. Prove that each diagonal of a rhombus is an Axis of Symmetry of the rhombus.

4. Take the midpoint on each side of an equilateral triangle; the segments joining them form a second equilateral triangle. [See the final chapter for a solution.]

5. Prove [Cor. 1.5.1].

PROPOSITION 1.6. *ISOSCELES TRIANGLES II.*

If a given triangle has two equal angles, then the sides opposite the two angles are equal in length (i.e., the triangle is isosceles).

PROOF. Let $\triangle ABC$ be a triangle constructed such that $\angle ABC = \angle ACB$. In order to prove our claim that sides $AB = AC$, we will use a proof by contradiction[21].

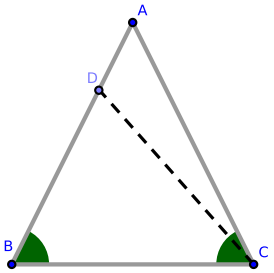


FIGURE 1.4.7. [1.6]

Without loss of generality[22], suppose side $AB > AC$. On AB, construct a point D such that $\overline{BD} = \overline{CA}$ [1.3] and construct $\overline{CD}$. Notice that $\triangle ACD > 0$.

Consider $\triangle DBC$ and $\triangle ACB$: since $DB = AC$, $\angle DBC = \angle ACB$, and each triangle contains the side BC, by [1.4] we have that $\triangle DBC \cong \triangle ACB$. It follows that $\triangle ACB = \triangle DBC$.

But $\triangle ACB = \triangle ACD \oplus \triangle DBC$, and so $\triangle ACD = 0$ and $\triangle ACD > 0$, a contradiction.

Therefore, we must have that $AB = AC$, and our claim has been proved. □

COROLLARY. *1.6.1. Taken together, [1.5] and [1.6] state that a triangle is isosceles if and only if*[23] *the angles at its base are equal.*

---

[21] That is, we will show that the statement "A triangle with two equal angles has unequal opposite sides" is false.
See also: `https://en.wikipedia.org/wiki/Proof_by_contradiction`

[22] This term is used before an assumption in a proof which narrows the premise to some special case; it is implied that either the proof for that case can be easily applied to all others or that all other cases are equivalent. Thus, given a proof of the conclusion in the special case, it is trivial to adapt it to prove the conclusion in all other cases. It is usually abbreviated as *wlog*; this textbook will use the abbreviation from this point forward.
See also: `http://en.wikipedia.org/wiki/Without_loss_of_generality`

[23] Often abbreviated as "iff".
See also: `https://en.wikipedia.org/wiki/If_and_only_if`

Exam questions.

1. What is the hypothesis in this proposition?
2. What proposition is this the converse of?
3. What is the inverse of this proposition?
4. What is the inverse of [1.5]?
5. What is meant by a proof by contradiction?
6. How does Euclid generally prove converse propositions?
7. What false assumption is made in order to prove the proposition?
8. What does this false assumption lead to?

Exercises.

1. Prove [Cor. 1.6.1].

PROPOSITION 1.7. *UNIQUENESS OF TRIANGLES.*

Suppose that we have two distinct triangles which share a common base. Also suppose that at one endpoint of the base we have sides of two triangles which share this vertex as an endpoint and are equal in length. It follows that the lengths of the sides connected to the other endpoint of the base are unequal in length.

PROOF. Construct distinct triangles $\triangle ADB$, $\triangle ACB$ which share the base AB. Suppose that $AC = AD$. We claim that $BC \neq BD$.

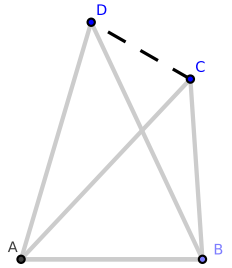


FIGURE 1.4.8. [1.7], case 1

The vertex of the second triangle may be within the first triangle or outside of the first triangle.

Case 1: *Vertex outside of the other triangle.*

Let the vertex of each triangle lie outside the interior of the other triangle (i.e., D does not lie inside $\triangle ACB$ and C does not lie inside $\triangle ADB$). Construct $\overline{CD}$. Because $AD = AC$ by hypothesis, $\triangle ACD$ is isosceles. By [Cor. 1.6.1], $\angle ACD = \angle ADC$.

Since $\angle ADC = \angle ADB + \angle BDC$ and $\angle ADB > 0$, it follows that $\angle ADC > \angle BDC$ [1.3.1 Axiom 12]. Because $\angle ACD = \angle ADC$, we also have that $\angle ACD > \angle BDC$. Since $\angle BCD = \angle BCA + \angle ACD$, we also have that $\angle BCD > \angle BDC$.

Consider $\triangle BDC$. Since $\angle BCD > \angle BDC$, by [Cor. 1.6.1] we have that $BD \neq BC$.

Case 2: *Vertex inside the other triangle.*

Wlog, let the vertex of the triangle $\triangle ADB$ be found within the interior of $\triangle ACB$. Extend side AC to segment $\overline{AE}$ and side AD to segment $\overline{AF}$. Construct $\overline{CD}$. Because $AC = AD$ by hypothesis, the triangle $\triangle ACD$ is isosceles; by [1.5] $\angle ECD = \angle FDC$.

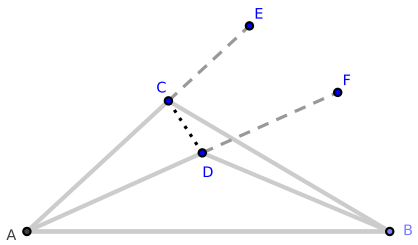


FIGURE 1.4.9. [1.7], case 2

Since $\angle ECD = \angle ECB + \angle BCD$, it follows that $\angle ECD > \angle BCD$ [1.3.1 Axiom 12]. Because $\angle ECD = \angle FDC$, we also have that $\angle FDC > \angle BCD$. Since $\angle BDC = \angle BDF + \angle FDC$, we have that $\angle BDC > \angle BCD$.

Consider $\triangle BDC$. Since $\angle BDC > \angle BCD$, by [Cor. 1.6.1] we have that $BD \neq BC$ □

COROLLARY. *1.7.1. Two triangles are distinct whenever no side of one triangle is equal in length to any side of the other triangle.*

Exercises.

1. Prove [Cor. 1.7.1].

PROPOSITION 1.8. *THE "SIDE-SIDE-SIDE" THEOREM FOR THE CONGRUENCE OF TRIANGLES.*

If all three pairs of sides of two triangles are equal in length, then the triangles are congruent.

PROOF. Suppose $\triangle ABC$ and $\triangle DEF$ are triangles where sides $AB = DE$, $AC = DF$, and $BC = EF$ (where BC and EF are the bases of those triangles). We claim that $\triangle ABC \cong \triangle DEF$.

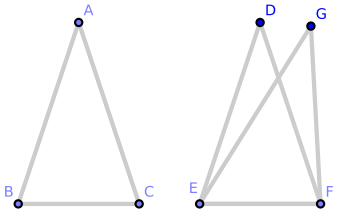


FIGURE 1.4.10. [1.8]

Let $\triangle ABC$ be positioned on $\triangle DEF$ such that point B coincides with point E and the side BC coincides with the side EF. Because $BC = EF$, the point C coincides with point F. If the vertex A falls on the same side of EF as vertex D, then the point A must coincide with D.

If this were not true, then A must have a different location: call this point G. Our hypothesis then states $EG = AB$ and $AB = ED$. By [1.3.1 Axiom 8], $EG = ED$. Similarly, $FG = FD$. However, by [1.7], $FG \neq FD$, a contradiction.

Hence the point A must coincide with point D, and so the three angles of one triangle are respectively equal to the three angles of the other (specifically, $\angle ABC = \angle DEF$, $\angle BAC = \angle EDF$, and $\angle BCA = \angle EFD$). Therefore, $\triangle ABC \cong \triangle DEF$. □

Exam questions.

1. What use is made of [1.7]? (Ans: As a lemma to [1.8].)

PROPOSITION 1.9. *BISECTING A RECTILINEAR ANGLE.*

It is possible to bisect an angle.

PROOF. Construct $\overrightarrow{AB}$, and take any point D on the ray $\overrightarrow{AB}$. Construct $\overrightarrow{AC}$, and take the point E on $\overrightarrow{AC}$ such that $\overline{AE} = \overline{AD}$ [1.3]. Construct $\overline{DE}$; also construct the equilateral triangle $\triangle DEF$ [1.1] such that F is on the other side of $\overline{DE}$ than A. Construct $\overline{AF}$. We claim that $\overline{AF}$ bisects $\angle BAC$.

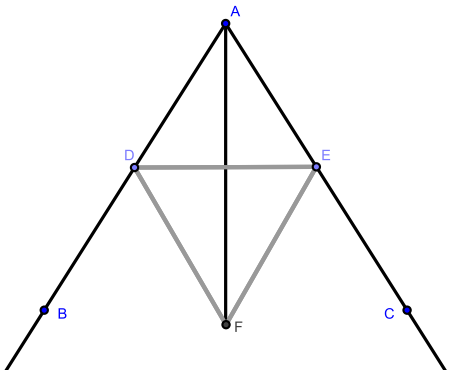


FIGURE 1.4.11. [1.9]

Notice that triangles $\triangle DAF$, $\triangle EAF$ share side AF. Given that $AD = AE$ by construction, we have that sides DA and AF of $\triangle DAF$ are respectively equal to sides EA and AF of $\triangle EAF$. Also, $DF = EF$ because they are the sides of an equilateral triangle [Def. 1.21].

By [1.8], $\triangle DAF \cong \triangle EAF$, and so $\angle DAF = \angle EAF$. Since

$$\begin{aligned} \angle BAC &= \angle DAF + \angle EAF \\ &= 2 \cdot \angle DAF \end{aligned}$$

then

$$\frac{1}{2} \cdot \angle BAC = \angle DAF$$

Or, $\angle BAC$ is bisected by $\overline{AF}$.

□

COROLLARY. *1.9.1. If $\overline{AF}$ is extended to the line $\overleftrightarrow{AF}$, then $\overleftrightarrow{AF}$ is the Axis of Symmetry of the $\triangle AED$, $\triangle DEF$, figure $BDAEC$, and segment $\overline{DE}$.*

COROLLARY. *1.9.2. In [1.9], AB and AC may be constructed as lines, rays, or segments of appropriate length with point A as the vertex, mutatis mutandis.*

Exam questions.

1. Why does Euclid construct the equilateral triangle on the side opposite of A?

2. If the equilateral triangle were constructed on the other side of DE, in what case would the construction fail?

Exercises.

1. Prove [1.9] without using [1.8]. (Hint: use [1.5, #2].)

2. Prove that $\overline{AF} \perp \overline{DE}$. (Hint: use [1.5, #2].) [See the final chapter for a solution.]

3. Prove that any point on $\overline{AF}$ is equally distant from the points D and E. [See the final chapter for a solution.]

4. Prove [Cor. 1.9.1].

5. Prove [Cor. 1.9.2].

PROPOSITION 1.10. *BISECTING A STRAIGHT-LINE SEGMENT.*

It is possible to bisect a segment of arbitrary length; i.e., it is possible to locate the midpoint of a segment.

PROOF. Construct the segment $\overline{AB}$; we claim that the segment $\overline{AB}$ can be bisected.

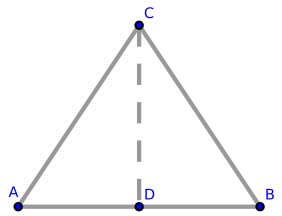


FIGURE 1.4.12. [1.10]

Construct the equilateral triangle $\triangle ABC$ with AB as its base [1.1]. Bisect $\angle ACB$ by constructing the segment $\overline{CD}$ [1.9] which intersects AB at point D. Clearly, $\overline{AB} = \overline{AD} \oplus \overline{DB}$. We claim that $\overline{AB} = 2 \cdot \overline{AD} = 2 \cdot \overline{DB}$. (This is equivalent to the idea that $\overline{AB}$ is bisected at D.)

Consider the two triangles $\triangle ACD$, $\triangle BCD$: notice that $AC = BC$ (since each are sides of the equilateral triangle $\triangle ACB$), and that these triangle have side CD in common. We also have that $\angle ACD = \angle BCD$ by construction. By [1.4], $\triangle ACD \cong \triangle BCD$, and so $\overline{AD} = \overline{DB}$. Therefore,

$$\overline{AB} = \overline{AD} + \overline{DB} = 2 \cdot \overline{AD} = 2 \cdot \overline{DB}$$

which proves our claim. □

Exercises.

1. Bisect a segment by constructing two circles. [See the final chapter for a solution.]

2. Extend $\overline{CD}$ to $\overleftrightarrow{CD}$. Prove that every point equally distant from the points A and B are points on $\overleftrightarrow{CD}$. [See the final chapter for a solution.]

PROPOSITION 1.11. *CONSTRUCTING A PERPENDICULAR STRAIGHT-LINE SEGMENT TO A LINE I.*

It is possible to construct a segment at a right angle to a given line from any point on the line.

PROOF. Construct the line $\overleftrightarrow{AB}$ containing the point C. On $\overrightarrow{CA}$, choose any point D; on $\overrightarrow{CB}$, choose the point E such that $\overline{CE} = \overline{CD}$ [1.3]. Construct the equilateral triangle $\triangle DFE$ on $\overline{DE}$ [1.1] and construct $\overline{CF}$. We claim that $\overleftrightarrow{AB} \perp \overline{CF}$.

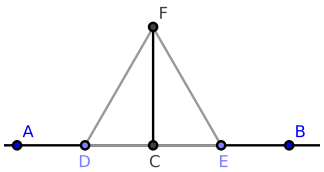


FIGURE 1.4.13. [1.11]

Consider the two triangles $\triangle DCF$ and $\triangle ECF$: they share side CF, $CD = CE$, and $DF = EF$ since $\triangle DFE$ is equilateral. By [1.8] $\triangle DCF \cong \triangle ECF$, and so $\angle DCF = \angle ECF$. [Def. 1.13] states that each of these angles is a right angle; this proves our claim. □

COROLLARY. *1.11.1. [1.11] holds when AB is a segment or ray and/or when CF is a straight line or a ray, mutatis mutandis.*

Exercises.

1. Prove that the diagonals of a rhombus bisect each other perpendicularly. [See the final chapter for a solution.]

2. Prove [1.11] without using [1.8].

3. Find a point on a given line that is equally distant from two given points. [See the final chapter for a solution.]

4. Find a point on a given line such that if it is joined to two given points on opposite sides of the line, then the angle formed by the connecting segment is bisected by the given line. (Hint: similar to the proof of #3.)

Definition.

38. The circumcenter[24] of a triangle is the point where the three perpendicular bisectors of a triangle intersect.

5. Find a point that is equidistant from three given points. (Hint: you are looking for the circumcenter of the triangle.) [See the final chapter for a solution.]

6. Prove [Cor. 1.11.1].

[24]See also: `http://www.mathopenref.com/trianglecircumcenter.html`

PROPOSITION 1.12. *CONSTRUCTING A PERPENDICULAR STRAIGHT-LINE SEGMENT TO A LINE II.*

Given an arbitrary straight line and an arbitrary point not on the line, we may construct a perpendicular segment from the point to the line.

PROOF. Construct $\overleftrightarrow{AB}$ and point C such that C is not on $\overleftrightarrow{AB}$. We wish to construct $\overline{CH}$ such that H is a point on $\overleftrightarrow{AB}$ and $\overline{CH} \perp \overleftrightarrow{AB}$.

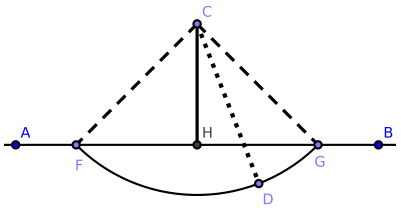


FIGURE 1.4.14. [1.12]

Take any point D from the opposite side of $\overleftrightarrow{AB}$ to C. Construct the circle $\bigcirc C$ with $\overline{CD}$ as its radius [Postulate 1.3] where $\bigcirc C$ intersects $\overleftrightarrow{AB}$ at the points F and G. Bisect $\overline{FG}$ at H [1.10] and construct $\overline{CH}$ [Postulate 1.1]. We claim that $\overline{CH} \perp \overleftrightarrow{AB}$.

Construct $\triangle CFG$: then the triangles $\triangle FHC$ and $\triangle GHC$ have sides $FH = GH$ by construction and side HC in common. Also, $CF = CG$ since both are radii of $\bigcirc C$ [Def. 1.32]. By [1.8], $\triangle FHC \cong \triangle GHC$, and so $\angle CHF = \angle CHG$. Since these are adjacent angles, [Def. 1.13] states that each is a right angle. This proves our claim. □

COROLLARY. *1.12.1. [1.12] holds when CH and/or AB are replaced by rays, mutatis mutandis.*

Exercises.

1. Prove that circle $\bigcirc C$ cannot meet $\overleftrightarrow{AB}$ at more than two points. [See the final chapter for a solution.]

2. Prove [Cor. 1.12.1].

PROPOSITION 1.13. *ANGLES AT INTERSECTIONS OF STRAIGHT LINES.*

If a line intersects another line, the lines will either make two right angles or two angles whose sum equals two right angles.

PROOF. If the line $\overleftrightarrow{AB}$ intersects the line $\overleftrightarrow{CD}$ at B, we claim that either $\angle ABC$ and $\angle ABD$ are right angles or that $\angle ABC + \angle ABD$ equals two right angles.

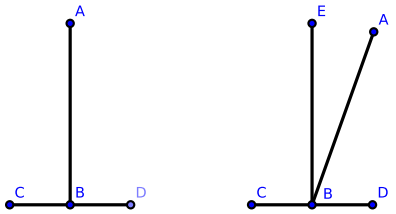


FIGURE 1.4.15. [1.13] (α) on left, (β) on right

If $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ as in Fig. 1.4.15(α), then $\angle ABC$ and $\angle ABD$ are right angles.

Otherwise, $\angle ABC$ and $\angle ABD$ are not right angles as in Fig. 1.4.15(β); construct $\overleftrightarrow{BE} \perp \overleftrightarrow{CD}$ [1.11]. Notice that $\angle ABC = \angle CBE + \angle EBA$ [Def. 1.11]. Adding $\angle ABD$ to each side of this equality, we obtain that

$$\angle ABC + \angle ABD = \angle CBE + \angle EBA + \angle ABD$$

Similarly, we have that

$$\angle CBE + \angle EBA + \angle ABD = \angle CBE + \angle EBD$$

By [1.3.1 Axiom 8], we have that

$$\angle ABC + \angle ABD = \angle CBE + \angle EBD$$

Since $\angle CBE$ and$\angle EBD$ are right angles, $\angle ABC + \angle ABD$ equals the sum of two right angles. This proves our claim. □

An alternate proof:

PROOF. Denote $\angle EBA$ by θ. We then have that

$$\begin{aligned} \angle CBA &= \text{right angle } + \theta \\ \angle ABD &= \text{right angle } - \theta \quad \Rightarrow \\ \angle CBA + \angle ABD &= \text{right angle} \end{aligned}$$

□

COROLLARY. *1.13.1. The above proposition holds when the straight lines are replaced by segments and/or rays, mutatis mutandis.*

COROLLARY. *1.13.2. The sum of two supplemental angles equals two right angles.*

COROLLARY. *1.13.3. Two distinct straight lines cannot share a common segment.*

COROLLARY. *1.13.4. The bisector of any angle bisects the corresponding reentrant angle.*

COROLLARY. *1.13.5. The bisectors of two supplemental angles are at right angles to each other.*

COROLLARY. *1.13.6. The angle* $\angle EBA = \frac{1}{2} \cdot (\angle CBA - \angle ABD)$.

Exercises.

1. Prove [Cor. 1.13.1].
2. Prove [Cor. 1.13.2].
3. Prove [Cor. 1.13.3].
4. Prove [Cor. 1.13.4].
5. Prove [Cor. 1.13.5].
6. Prove [Cor. 1.13.6].

PROPOSITION 1.14. *RAYS TO STRAIGHT LINES.*

If at the endpoint of a ray there exists two other rays constructed on opposite sides of the first ray such that the sum of their adjacent angles is equal to two right angles, then these two rays form one line.

PROOF. Construct $\overrightarrow{BA}$. On opposite sides of $\overrightarrow{BA}$, construct $\overrightarrow{BC}$ and $\overrightarrow{BD}$ such that the sum of their adjacent angles, $\angle CBA + \angle ABD$, equals two right angles. We claim that $\overrightarrow{BC} \oplus \overrightarrow{BD} = \overleftrightarrow{CD}$.

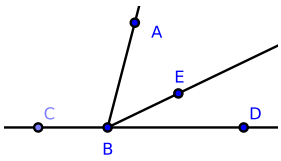


FIGURE 1.4.16. [1.14]

Suppose instead that $\overrightarrow{BC} \oplus \overrightarrow{BE} = \overleftrightarrow{CE}$ and $\angle EBD > 0$. Since $\overleftrightarrow{CE}$ is a line and $\overrightarrow{BA}$ stands on it, the sum $\angle CBA + \angle ABE$ equals two right angles [1.13]. Also by hypothesis, the sum $\angle CBA + \angle ABD$ equals two right angles. Therefore,

$$\begin{aligned} \angle CBA + \angle ABE &= \angle CBA + \angle ABD \\ \angle ABE &= \angle ABD \\ \angle ABE &= \angle ABE + \angle EBD \\ 0 &= \angle EBD \end{aligned}$$

Since$\angle EBD > 0$, we have a contradiction. Hence, $\overrightarrow{BC} \oplus \overrightarrow{BD} = \overleftrightarrow{CD}$, which proves our claim. □

COROLLARY. *1.14.1. The above result holds for segments, mutatis mutandis.*

Exercises.

1. Prove [Cor. 1.14.1].

PROPOSITION 1.15. *OPPOSITE ANGLES ARE EQUAL.*

If two lines intersect at a point, then their opposite angles are equal.

PROOF. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E, then we claim that $\angle AEC = \angle DEB$ and $\angle BEC = \angle DEA$.

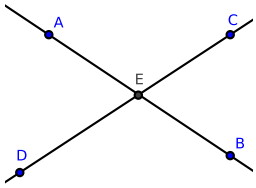


FIGURE 1.4.17. [1.15]

Because $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ at E, the sum $\angle AEC + \angle DEA$ equals two right angles [1.13]. Similarly, because $\overleftrightarrow{CD}$ intersects $\overleftrightarrow{AB}$ at E, the sum $\angle AEC + \angle BEC$ also equals two right angles. Therefore,

$$\begin{aligned} \angle AEC + \angle DEA &= \angle AEC + \angle BEC \\ \angle DEA &= \angle BEC \end{aligned}$$

Similarly, we can also show that $\angle AEC = \angle DEB$, and the claim is proved. □

An alternate proof:

PROOF. Because opposite angles share a common supplement, they are equal. □

COROLLARY. *1.15.1. [1.15] holds when either one or both of the two straight lines are replaced either by segments or by rays, mutatis mutandis.*

Exam questions for [1.13]–[1.15].

1. What problem is required in Euclid's proof of [1.13]?
2. What theorem? (Ans. No theorem, only the axioms.)
3. If two lines intersect, how many pairs of supplemental angles do they make?
4. What is the relationship between [1.13] and [1.14]?
5. What three lines in [1.14] are concurrent?
6. State the converse of Proposition [1.15] and prove it.
7. What is the subject of [1.13], [1.14], [1.15]? (Ans. Angles at a point.)

Exercises.

1. Prove [Cor. 1.15.1].

PROPOSITION 1.16. *THE EXTERIOR ANGLE OF A TRIANGLE IS GREATER THAN EITHER OF THE NON-ADJACENT INTERIOR ANGLES.*

If any side of a triangle is extended, the resulting exterior angle is greater than either of the non-adjacent interior angles.

PROOF. Construct $\triangle ABC$. Wlog, we extend side BC to the segment $\overline{BD}$. We claim that the exterior angle $\angle ACD$ is greater than either of the interior non-adjacent angles $\angle ABC$, $\angle BAC$.

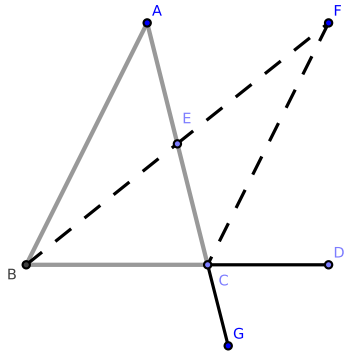


FIGURE 1.4.18. [1.16]

Bisect AC at E [1.10] and construct $\overline{BE}$ [Postulate 1.1]. Extend $\overline{BE}$ to $\overline{BF}$ such that $\overline{BE} = \overline{EF}$ [1.3]. Construct $\overline{CF}$.

Consider $\triangle CEF$ and $\triangle AEB$: $\overline{CE} = \overline{EA}$ by construction, and so sides CE, EF in $\triangle CEF$ are respectively equal in length to sides EA, EB in $\triangle AEB$; also, $\angle CEF = \angle AEB$ [1.15]. By [1.4], $\triangle CEF \cong \triangle AEB$, and so $\angle ECF = \angle EAB$. However, $\angle ACD > \angle ECF$ since $\angle ACD = \angle ECF + \angle FCD$. Therefore, the angle $\angle ACD > \angle EAB = \angle BAC$ (claim 1)

Similarly, if BC is bisected, it can be shown that $\angle BCG = \angle ACD > \angle ABC$ (claim 2), which completes the proof. □

COROLLARY. *1.16.1. The sum of the three interior angles of the triangle $\triangle BCF$ is equal to the sum of the three interior angles of the triangle $\triangle ABC$.*

COROLLARY. *1.16.2. The area of $\triangle BCF$ is equal to the area of $\triangle ABC$, which we will write as $\triangle BCF = \triangle ABC$.*

COROLLARY. *1.16.3. If sides BA and CF are extended to lines, they cannot meet at any finite distance: if they met at any point X, then the triangle $\triangle CAX$ would have an exterior angle $\angle BAC$ equal to the interior angle $\angle ACX$.*

Exercise.

1. Prove [Cor. 1.16.1].
2. Prove [Cor. 1.16.2].
3. Prove [Cor. 1.16.3] using a proof by contradiction.

PROPOSITION 1.17. *THE SUM OF TWO INTERIOR ANGLES OF A TRIANGLE.*

The sum of two interior angles of a triangle is less than the sum of two right angles.

PROOF. We claim that the sum of any two interior angles of $\triangle ABC$ is less than the sum of two right angles.

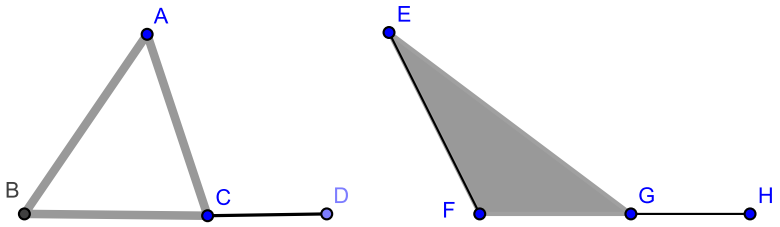


FIGURE 1.4.19. [1.17]

Wlog, choose $\angle ABC$ and $\angle BAC$ and extend side BC to the segment $\overline{BD}$. By [1.16], $\angle ACD > \angle ABC$. To each, add the angle $\angle ACB$:

$$\angle ACD + \angle ACB \quad > \quad \angle ABC + \angle ACB$$

By [1.13], $\angle ACD + \angle ACB$ equals two right angles; therefore, $\angle ABC + \angle ACB$ is less than two right angles.

Similarly, we can show that the sums $\angle ABC + \angle BAC$ and $\angle ACB + \angle BAC$ are each less than two right angles. A similar argument follows on $\triangle EFG$, *mutatis mutandis*, which proves our claim. □

COROLLARY. *1.17.1. Every triangle has at least two acute angles.*

COROLLARY. *1.17.2. If two angles of a triangle are unequal, then the shorter angle is acute.*

Exercises.

1. Prove [1.17] without extending a side. (Attempt after completing Chapter 1. Hint: use parallel line theorems.)

2. Prove [Cor. 1.17.1].

3. Prove [Cor. 1.17.2].

PROPOSITION 1.18. *ANGLES AND SIDES IN A TRIANGLE I.*

If one side of a triangle is longer than another side, then the angle opposite to the longer side is greater in measure than the angle opposite to the shorter side.

PROOF. Construct $\triangle ABC$ with sides AB and AC where $AC > AB$. We claim that the angle opposite AC ($\angle ABC$) is greater in measure than the angle opposite AB ($\angle ACB$).

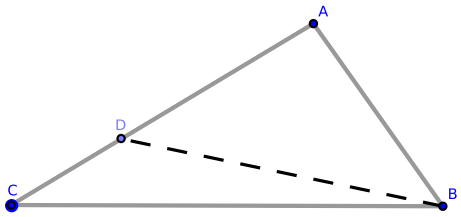


FIGURE 1.4.20. [1.18]

On AC, find D such that $\overline{AD} = \overline{AB}$ [1.3], and construct $\overline{BD}$; notice that $\triangle ABD$ is isosceles. Hence, $\angle ADB = \angle ABD$. Since $\angle ADB > \angle ACB$ by [1.16], $\angle ABD > \angle ACB$. Since $\angle ABC = \angle ABD + \angle CBD$, we also have that $\angle ABC > \angle ACB$, which proves our claim. □

An alternate proof:

PROOF. Suppose we have $\triangle ABC$ with sides $AC > AB$. We claim that $\angle ABC > \angle ACB$.

With A as the center and with the shorter side AB as radius, construct the circle $\bigcirc A$ which intersects AC at D and BC at E.

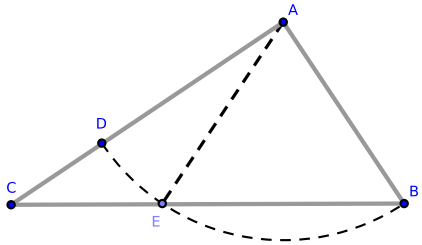


FIGURE 1.4.21. [1.18], alternate proof

Construct $\overline{AE}$. Since $\overline{AB} = \overline{AE}$, $\angle AEB = \angle ABE$ [Cor. 1.6.1]. Notice that $\angle ABE = \angle ABC$, and so $\angle AEB = \angle ABC$. By [1.16], $\angle AEB > \angle ACB$; therefore, $\angle ABC > \angle ACB$, which proves our claim. □

Exercises.

1. Prove that if two of the opposite sides of a quadrilateral are respectively the greatest and the least sides of the quadrilateral, then the angles adjacent to the least are greater than their opposite angles.

2. In any triangle, prove that the perpendicular from the vertex opposite the side which is not less than either of the remaining sides falls within the triangle.

PROPOSITION 1.19. *ANGLES AND SIDES IN A TRIANGLE II.*

In a triangle, if one angle is greater in measure than another, then the side opposite the greater angle is longer than the side opposite the shorter angle.

PROOF. Construct $\triangle ABC$ where $\angle ABC > \angle ACB$. We claim that $AC > AB$.

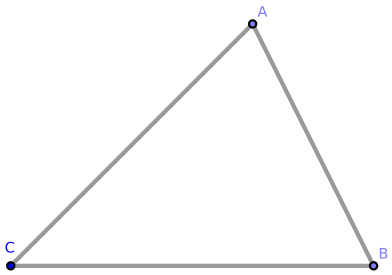


FIGURE 1.4.22. [1.19]

If $AC \ngtr AB$, then either $AC = AB$ or $AC < AB$.

1. If $AC = AB$, $\triangle ACB$ is isosceles and $\angle ACB = \angle ABC$ [Cor. 1.6.1]. This contradicts our hypothesis that $\angle ABC > \angle ACB$, and so $AC \neq AB$.

2. If $AC < AB$, we have that $\angle ACB < \angle ABC$ [1.18]. This also contradicts our hypothesis that $\angle ABC > \angle ACB$, and so $AC \not< AB$.

Since $AC \not\leq AB$, we must have that $AC > AB$. □

COROLLARY. *1.19.1. In a triangle, greater (lesser) sides stand opposite the greater (lesser) angles and greater (lesser) angles stand opposite the greater (lesser) sides.*

Exercises.

1. Prove this proposition by a direct demonstration.

2. Prove that a segment from the vertex of an isosceles triangle to any point on the base is less than either of the equal sides but greater if the base is extended and the point of intersection falls outside of the triangle.

3. Prove that three equal and distinct segments cannot be constructed from the same point to the same line. [See the final chapter for a solution.]

4. Consider [1.16], Fig 1.4.18: if AB is the longest side of the $\triangle ABC$, then BF is the longest side of $\triangle FBC$ and $\angle BFC < \frac{1}{2} \cdot \angle ABC$.

5. If $\triangle ABC$ is a triangle such that side $AB \leq AC$, then a segment $\overline{AG}$, constructed from A to any point G on side BC, is less than AC. [See the final chapter for a solution.]

6. Prove [Cor. 1.19.1].

PROPOSITION 1.20. *THE SUM OF THE LENGTHS OF ANY PAIR OF SIDES OF A TRIANGLE.*

In a triangle, the sum of the lengths of any pair of sides is greater than the length of the remaining side.

PROOF. Construct $\triangle ABC$; wlog, we claim that $AB + AC > BC$.

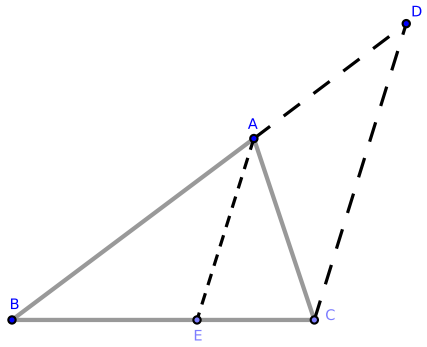


FIGURE 1.4.23. [3.20]

Extend BA to the segment $\overline{BD}$ such that $\overline{AD} = \overline{AC}$ [1.3]. Construct $\overline{CD}$.

Consider$\triangle ACD$: by construction $\overline{AD} = \overline{AC}$, and so $\angle ACD = \angle ADC$ [1.5]. Since $\angle BCD = \angle BCA + \angle ACD$, $\angle BCD > \angle ACD = \angle ADC = \angle BDC$. By [1.19], $\overline{BD} > \overline{BC}$.

Notice that

$$\begin{aligned} \overline{AD} &= \overline{AC} \\ \overline{BA} + \overline{AD} &= \overline{BA} + \overline{AC} \\ \overline{BD} &= \overline{BA} + \overline{AC} \end{aligned}$$

and so $\overline{BA} + \overline{AC} > \overline{BC}$. This proves our claim. □

Alternatively:

PROOF. Construct $\triangle ABC$; wlog, we claim that $AB + AC > BC$.

Bisect $\angle BAC$ by constructing $\overline{AE}$ [1.9]. Then $\angle BEA > \angle EAC = \angle EAB$. It follows that $\overline{BA} > \overline{BE}$ [1.19]. Similarly, $\overline{AC} > \overline{EC}$. It follows that $\overline{BA} + \overline{AC} > \overline{BE} + \overline{EC} = \overline{BC}$. □

Exercises.

1. Let a, b, and c be side-lengths of any triangle. Prove that

$$|a - b| < c < (a + b)$$

2. Any side of any polygon is less than the sum of the remaining sides.

3. The perimeter of any triangle is greater than the perimeter of any inscribed triangle and less than the perimeter of any circumscribed triangle. (See also [Def. 4.1].)

4. The perimeter of any polygon is greater than that of any inscribed (and less than that of any circumscribed) polygon of the same number of sides.

5. The perimeter of a quadrilateral is greater than the sum of its diagonals. [See the final chapter for a solution.]

6. The sum of the lengths of the three medians of a triangle is less than $\frac{3}{2}$ times its perimeter. [See the final chapter for a solution.]

PROPOSITION 1.21. *TRIANGLES WITHIN TRIANGLES.*

In an arbitrary triangle, if two segments are constructed from the vertexes of its base to a point within the triangle, then:

(1) the sum of the lengths of these inner sides will be less than the sum of the outer sides (excluding the base);

(2) these inner sides will contain a greater angle than the corresponding sides of the outer triangle.

PROOF. Construct $\triangle ABC$ with base BC, and construct point D within the interior of $\triangle ABC$. Finally, construct segments $\overline{BD}$, $\overline{CD}$. We claim that:

1. $BA + AC > BD + DC$
2. $\angle BDC > \angle BAC$

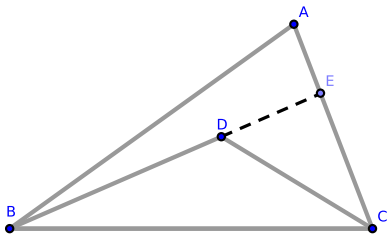


FIGURE 1.4.24. [3.21]

Claim 1: $BA + AC > BD + DC$.

Extend BD to $\overline{BE}$ where E is a point on AC. In $\triangle BAE$, we have that $BA + AE > BE$ [1.20]. It follows that:

$$BA + AC = BA + AE + EC > BE + EC$$

Similarly, in $\triangle DEC$, we have that $DE + EC > DC$, from which it follows that

$$BE + EC = BD + DE + EC > BD + DC$$

From these two inequalities, we obtain that $BA + AC > BD + DC$ (proving claim 1).

Claim 2: $\angle BDC > \angle BAC$.

By [1.16], we have that $\angle BDC > \angle BEC$. Similarly, $\angle BEC > \angle BAE$. It follows that $\angle BDC > \angle BAE = \angle BAC$. □

An alternative proof to claim 2 that does not extend sides BD and DC:

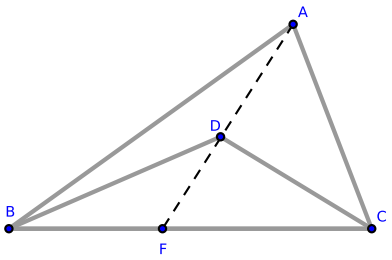


FIGURE 1.4.25. [3.21, alternate proof]

PROOF. Construct $\triangle ABC$ and $\triangle BDC$ as before, then constructing $\overline{AD}$ and extending it to intersect BC at F. Consider $\triangle BDA$ and $\triangle CDA$. By [1.16], $\angle BDF > \angle BAF$ and $\angle FDC > \angle FAC$. Since

$$\begin{aligned} \angle BDF + \angle FDC &= \angle BDC \\ \angle BAF + \angle FAC &= \angle BAC \end{aligned}$$

we have that $\angle BDC > \angle BAC$. □

Exercises.

1. The sum of the side lengths constructed from any point within a triangle to its vertices is less than the length of the triangle's perimeter.

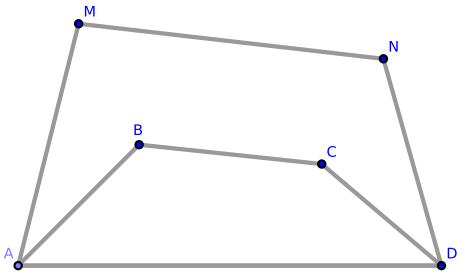


FIGURE 1.4.26. [1.21, #2]

2. If a convex polygonal line $ABCD$ lies within a convex polygonal line $AMND$ terminating at the same endpoints, prove that the length of $ABCD$ is less than that of $AMND$.

PROPOSITION 1.22. *CONSTRUCTION OF TRIANGLES FROM ARBITRARY SEGMENTS.*

It is possible to construct a triangle whose three sides are respectively equal in length to three arbitrary segments whenever the sum of the lengths of every two pairs of segments is greater than the length of the remaining segment.

PROOF. Let $\overline{AR}$, $\overline{BS}$, and $\overline{CT}$ be arbitrary segments which satisfy our hypothesis.

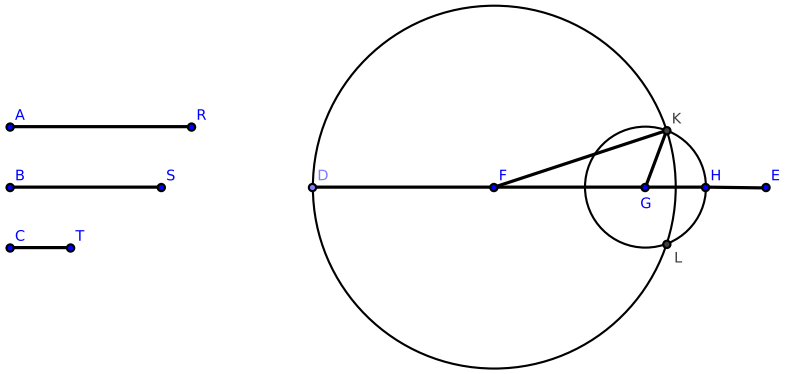


FIGURE 1.4.27. [1.22]

Construct $\overrightarrow{DE}$ such that it contains the segments $\overline{DF} = \overline{AR}$, $\overline{FG} = \overline{BS}$, and $\overline{GH} = \overline{CT}$ [1.3]. With F as the center and $\overline{DF}$ as radius, construct the circle $\bigcirc F$ [Section 1.2, Postulate 3]. With G as the center and $\overline{GH}$ as radius, construct the circle $\bigcirc G$ where K is one intersection between the circles $\bigcirc F$ and $\bigcirc G$. Construct $\overline{KF}$, $\overline{KG}$. We claim that $\triangle KFG$ is the required triangle.

Since F is the center of $\bigcirc F$, $\overline{FK} = \overline{FD}$. Since $\overline{FD} = \overline{AR}$ by construction, $\overline{FK} = \overline{AR}$. Also by construction, $\overline{FG} = \overline{BS}$ and $\overline{KG} = \overline{CT}$. Therefore, the three sides of the triangle $\triangle KFG$ are respectively equal to the three segments $\overline{AR}$, $\overline{BS}$, and $\overline{CT}$, which proves our claim. □

Exercises.

1. Prove that when the above conditions are fulfilled that the two circles must intersect.

2. If the sum of two of the segments equals the length of the third, prove that the segments will not intersect.

PROPOSITION 1.23. *CONSTRUCTING AN ANGLE EQUAL TO AN ARBITRARY RECTILINEAR ANGLE.*

It is possible to construct an angle equal to an arbitrary angle on an endpoint of a segment.

PROOF. Construct an arbitrary angle $\angle DEF$ from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. We claim it is possible to construct an angle equal to $\angle DEF$ on a given point A.

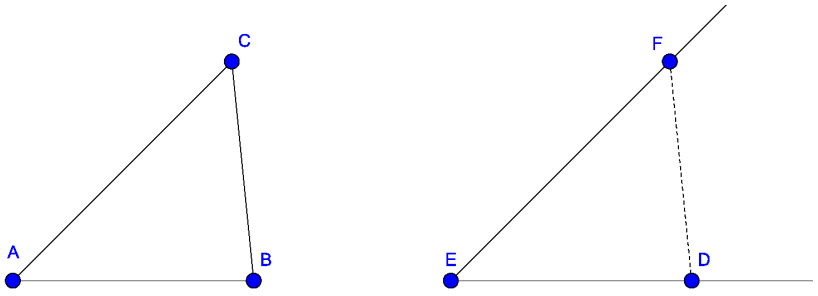


FIGURE 1.4.28. [1.23]

Construct $\overline{DF}$, and construct the triangle $\triangle BAC$ where $\overline{AB} = \overline{ED}$, $\overline{AC} = \overline{EF}$, and $\overline{CB} = \overline{FD}$ [1.22]. By [1.8], $\triangle BAC \cong \triangle DEF$, and so $\angle BAC = \angle DEF$. □

Exercises.

1. Construct a triangle given two sides and the angle between them. [See the final chapter for a solution.]

2. Construct a triangle given two angles and the side between them.

3. Construct a triangle given two sides and the angle opposite one of them.

4. Construct a triangle given the base, one of the angles at the base, and the sum or difference of the sides.

5. Given two points, one of which is in a given line, find another point on the given line such that the sum or difference of its distances from the former points may be given. Show that two such points may be found in each case.

PROPOSITION 1.24. *ANGLES AND SIDES IN A TRIANGLE III.*

If two pairs of sides in two triangles are respectively equal in length where the interior angle in one triangle is greater in measure than the interior angle of the other triangle, then the triangle with the greater interior angle will have a longer third side than the triangle with the smaller interior angle.

PROOF. Construct two triangles $\triangle ABC$ and $\triangle DEF$ where $AB = DE$, $AC = DF$, and $\angle BAC > \angle EDF$. We claim that $BC > EF$.

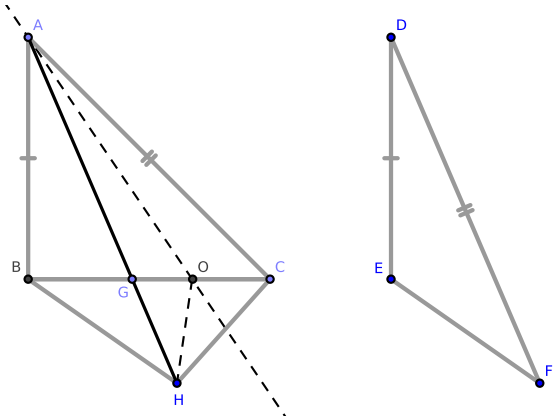


FIGURE 1.4.29. [1.24]

Construct point G on BC such that $\angle BAG = \angle EDF$. Wlog, suppose that $AB < AC$; by [1.19, #5] we have that $\overline{AG} < \overline{AC}$. Extend $\overline{AG}$ to $\overline{AH}$ where $\overline{AH} = \overline{DF} = \overline{AC}$ [1.3].

Construct $\overline{BH}$. In triangles $\triangle BAH$ and $\triangle EDF$, we have $AB = DE$, $AH = DF$, and $\angle BAH = \angle EDF$ by construction. By [1.4], it follows that $\triangle BAH \cong \triangle EDF$, and so $BH = EF$.

Notice that $\angle ACH > \angle BCH$ since $\angle ACH = \angle BCH + \angle BCA$. Because $AH = AC$ by construction, $\triangle ACH$ is isosceles; therefore, $\angle ACH = \angle AHC$ [1.5]. It follows that $\angle AHC > \angle BCH$. And since $\angle BHC = \angle BHA + \angle AHC$, we also have that $\angle BHC > \angle BCH$.

It follows that in $\triangle BCH$, we have $BC > BH$ by [1.19]. Since $BH = EF$, we have that $BC > EF$, which proves our claim. □

Alternatively, the concluding part of this proposition may be proved without constructing $\overline{CH}$.

PROOF. For simplicity's sake, we'll construct the triangles as above; we claim that our proof won't require $\overline{CH}$.

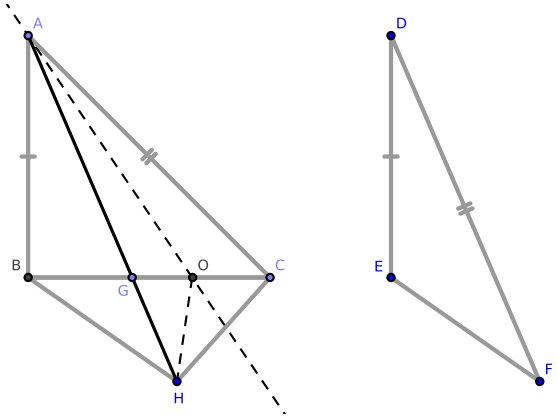


FIGURE 1.4.30. [1.24]

Notice that

$$\begin{array}{rcll} BG+GH & > & BH & [1.20] \quad \text{and} \\ AG+GC & > & AC & [1.20] \quad \Rightarrow \\ (BG+GH)+(AG+GC) & > & BH+AC & \quad \Rightarrow \\ (BG+GC)+(AG+GH) & > & BH+AC & \quad \Rightarrow \\ BC+AH & > & BH+AC & \end{array}$$

Since $AH = AC$ and $BH = EF$ by construction, we have that $BC > EF$, which proves our claim. □

Another alternative:

PROOF. In $\triangle ABC$, bisect the angle $\angle CAH$ by $\overleftrightarrow{AO}$. In $\triangle CAO$ and $\triangle HAO$ we have the sides CA, AO in one triangle respectively equal to the sides AH, AO in the other where the interior angles are equal. By [1.4], $OC = OH$. It follows that $BO + OH = BO + OC = BC$. But $BO + OH > BH$ [1.20]. Since $BH = EF$, $BC > EF$, which proves our claim. □

Exercises.

1. Prove this proposition by constructing the angle $\angle ABH$ to the left of AB.

2. Prove that the angle $\angle BCA > \angle EFD$.

PROPOSITION 1.25. *ANGLES AND SIDES IN A TRIANGLE IV.*

If in two triangles we have two pairs of sides in each triangle respectively equal to the other pair where the remaining sides of the triangles are unequal, then the triangle with the longer side will have a greater interior angle than the triangle with the shorter side.

PROOF. Construct $\triangle ABC$ and $\triangle DEF$ such that $AB = DE$, $AC = DF$, and $BC > EF$. We claim that $\angle BAC > \angle EDF$.

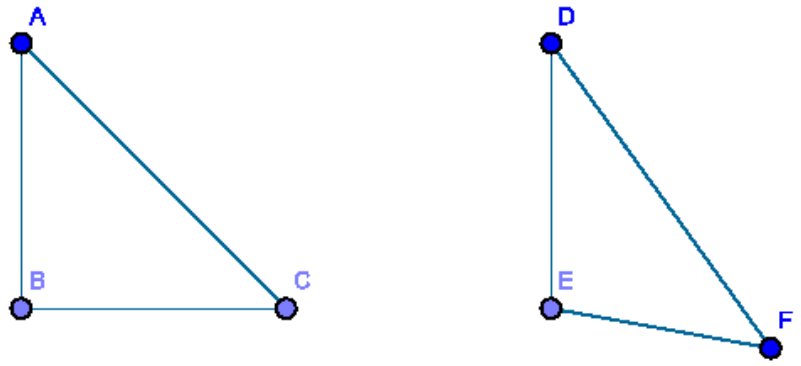


FIGURE 1.4.31. [1.25]

Suppose instead that $\angle BAC = \angle EDF$; since $AB = DE$ and $AC = DF$ by construction, by [1.4] $\triangle ABC \cong \triangle DEF$, and so $BC = EF$. This contradicts our hypothesis that $BC > EF$; hence, $\angle BAC \neq \angle EDF$.

Now suppose that $\angle BAC < \angle EDF$; since $AB = DE$ and $AC = DF$, by [1.24] we have that $EF > BC$. This contradicts our hypothesis that $BC > EF$; hence, $\angle EDF \ngtr \angle BAC$.

Therefore, $\angle BAC > \angle EDF$, which proves our claim. □

COROLLARY. *1.25.1. Construct $\triangle ABC$ and $\triangle DEF$ where $AB = DE$ and $AC = DF$. By [1.24] and [1.25], $BC < EF$ iff $\angle BAC < \angle EDF$.*

Exercise.

1. Demonstrate this proposition directly by constructing a segment on BC equal in length to EF.

PROPOSITION 1.26. *TWO ANGLES & ONE SIDE OF A TRIANGLE.*

If two triangles have two pairs of angles which are equal and two sides which are equal, then the triangles are congruent. Most textbooks express this result in two forms:

1. THE "ANGLE-SIDE-ANGLE" THEOREM FOR THE CONGRUENCE OF TRIANGLES. If the side in question is the side between the two angles, then the triangles are congruent.

2. THE "ANGLE-ANGLE-SIDE" THEOREM FOR THE CONGRUENCE OF TRIANGLES. If the side in question is not the side between the two angles, then the triangles are congruent.

PROOF. Construct $\triangle ABC$ and $\triangle DEF$ such that $\angle ABC = \angle DEF$ and $\angle ACB = \angle EFD$. We claim that if one side of $\triangle ABC$ is equal in length to its respective side in $\triangle DEF$, then $\triangle ABC \cong \triangle DEF$.

This proposition has two cases, which we will prove separately.

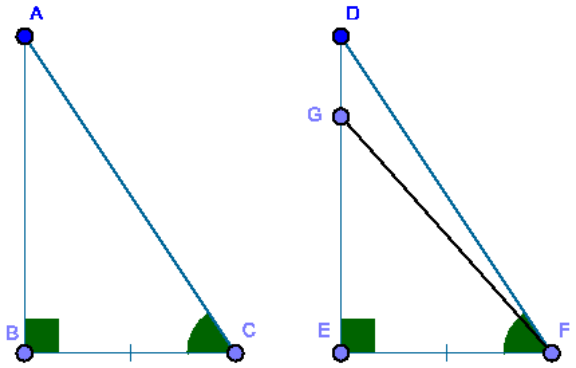


FIGURE 1.4.32. [1.26], case 1

Case 1. *ANGLE-SIDE-ANGLE*

Also suppose that $BC = EF$. If $AB \neq DE$, suppose that $AB = GE$. Construct $\overline{GF}$, and notice that $\angle GFD > 0$.

Consider $\triangle ABC$ and $\triangle GEF$: $AB = GE$, $BC = EF$, and $\angle ABC = \angle GEF$. By [1.4], $\triangle ABC \cong \triangle GEF$, and so $\angle ACB = \angle GFE$. Since $\angle ACB = \angle DFE$ by hypothesis, we have that $\angle GFE = \angle DFE$ and $\angle GFE + \angle GFD = \angle DFE$; hence $\angle GFD = 0$ and $\angle GFD > 0$, a contradiction. Therefore, $AB = DE$.

Since $AB = DE$, $BC = EF$, and $\angle ABC = \angle DEF$, by [1.4] $\triangle ABC \cong \triangle DEF$.

Case 2. *ANGLE-ANGLE-SIDE*

Now suppose that $AB = DE$. (The same result follows if $AC = DF$, *mutatis mutandis*.)

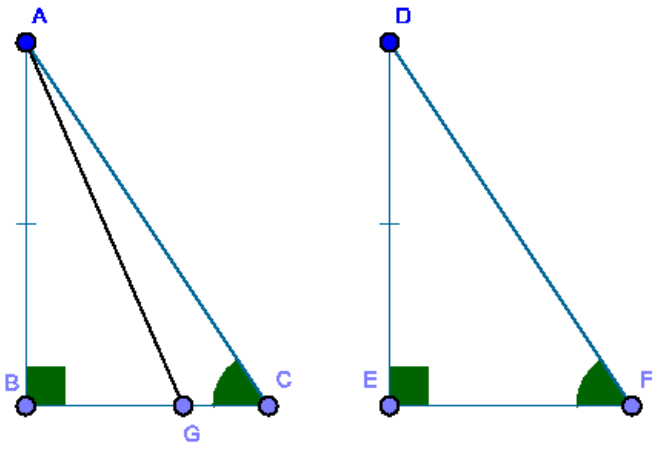


FIGURE 1.4.33. [1.26], case 2

If $BC \neq EF$, suppose that $EF = BG$. Construct $\overline{AG}$, and consider $\triangle ABG$ and $\triangle DEF$: $AB = DE$, $BG = EF$, and by hypothesis $\angle ABG = \angle DEF$. By [1.4], $\triangle ABG \cong \triangle DEF$, and so $\angle AGB = \angle EFD$. Since $\angle ACB = \angle EFD$ by hypothesis, $\angle AGB = \angle ACB$; that is, the exterior angle of the triangle $\triangle ACG$ is equal to an interior and non-adjacent angle, which contradicts [1.16]. Thus $BC = EF$.

Since $AB = DE$, $BC = EF$, and $\angle ABC = \angle DEF$, by [1.4] $\triangle ABC \cong \triangle DEF$.

This proves our claim. □

Case 1 (*ANGLE-SIDE-ANGLE)* may also be proved immediately by superposition: it is evident if $\triangle ABC$ is positioned on $\triangle DEF$ such that the point B coincides with point E and $\overline{BC}$ coincides with $\overline{EF}$ that, since $\overline{BC} = \overline{EF}$, point C coincides with point F. And since the angles $\angle ABC$, $\angle BCA$ are respectively equal to the angles $\angle DEF$, $\angle EFD$, the segments $\overline{BA}$, $\overline{CA}$ will coincide with $\overline{ED}$, $\overline{FD}$. Hence, $\triangle ABC \cong \triangle DEF$.

Exercises.

1. Prove that the endpoints of the base of an isosceles triangle are equally distant from any point on the perpendicular segment from the vertical angle on the base.

2. Prove that if the line which bisects the vertical angle of a triangle also bisects the base, then the triangle is isosceles. [See the final chapter for a solution.]

3. In a given straight line, find a point such that the perpendiculars from it on two given lines are equal. State also the number of solutions.

4. Prove that if two right triangles have hypotenuses of equal length and an acute angle of one is equal to an acute angle of the other, then they are congruent.

5. Prove that if two right triangles have equal hypotenuses and that if a side of one is equal in length to a side of the other, then the triangles they are congruent. (Note: this proves the special case of Side-Side-Angle congruency for right triangles.) [See the final chapter for a solution.]

Definition

39. The bisectors of the three internal angles of a triangle are concurrent, and their point of intersection is called the **incenter** of the triangle.

6. The bisectors of two external angles and the bisector of the third internal angle are concurrent.

7. Through a given point, construct a straight line such that perpendiculars on it from two given points on opposite sides are equal to each other.

8. Through a given point, construct a straight line intersecting two given lines which forms an isosceles triangle with them.

1.5. Propositions from Book I: 27-48

Additional definitions regarding parallel lines:

Parallel Lines.

40. If two straight lines in the same plane do not meet at any finite distance, they are said to be *parallel*. If rays or segments can be extended into lines which do not meet at any finite distance, they are also said to be *parallel*.

41. A *parallelogram* is a quadrilateral where both pairs of opposite sides are parallel.

42. The segment joining either pair of opposite angles of a quadrilateral is called a *diagonal*. See Fig. 1.5.1.

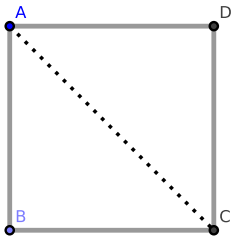


FIGURE 1.5.1. [Def 1.41] and [Def. 1.42]: $\overline{AC}$ is a diagonal of the square $\boxdot ABCD$, which is also a parallelogram.

45. A quadrilateral where one pair of opposite sides is parallel is called a *trapezoid*.

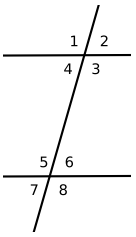


FIGURE 1.5.2. [Def. 1.46]

46. When a straight line intersects two other straight lines, between them are eight angles (see Fig. 1.5.2).

- Angles 1 and 2 are *exterior angles*; so are angles 7 and 8.
- Angles 3 and 4 are called *interior angle*s; so are angles 5 and 6.

- Angles 4 and 6 are called *alternate angles;* so are angles 3 and 5.
- Angles 1 and 5 are called *corresponding angles*; so are angles 2 and 6, 3 and 8, and 4 and 7.

These definitions hold when we replace straight lines with either rays or segments, *mutatis mutandis*.

PROPOSITION 1.27. *PARALLEL LINES I.*

If a line intersects two other lines and their alternate angles are equal, then these lines are parallel.

PROOF. Let $\overleftrightarrow{EF}$ intersect $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ such that their alternate angles are equal ($\angle AEF = \angle EFD$). We claim that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

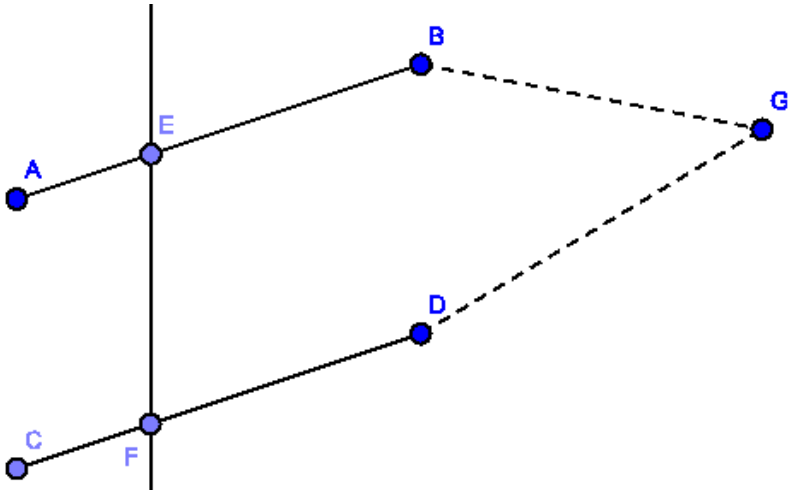


FIGURE 1.5.3. [1.27]

If $\overleftrightarrow{AB} \nparallel \overleftrightarrow{CD}$, then the lines must intersect at a point some finite distance from B: suppose they intersect at G.

It follows that $\triangle EGF$ is a triangle where $\angle AEF$ is an exterior angle and $\angle EFD$ a non-adjacent interior angle. By [1.16], $\angle AEF > \angle EFD$; but $\angle AEF = \angle EFD$ by hypothesis, a contradiction. Therefore, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. □

PROPOSITION 1.28. *PARALLEL LINES II & III.*

1) PARALLEL LINES II. If a line intersecting two lines makes the exterior angle equal to its corresponding interior angle, then the two lines are parallel.

2) PARALLEL LINES III. If a line intersecting two lines makes the sum of two interior angles on the same side equal to two right angles, the two lines are parallel.

PROOF. If $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ such that the exterior angle ($\angle EGB$) equals its corresponding interior angle ($\angle DHG$), or if it makes the sum of two interior angles on the same side ($\angle BGH$, $\angle GHD$) equal to two right angles, then we claim that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

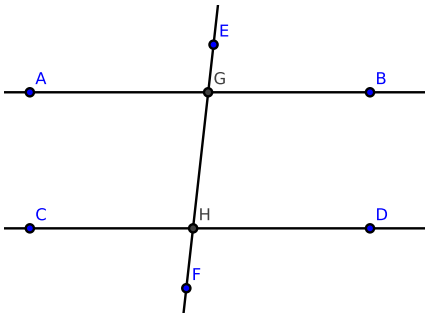


FIGURE 1.5.4. [1.28]

Claim 1: suppose that $\angle EGB = \angle DHG$. Since $\overleftrightarrow{AB}$, $\overleftrightarrow{EF}$ intersect at G, $\angle AGH = \angle EGB$ [1.15]. It follows that $\angle AGH = \angle DHG$. Since these are alternate angles, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ by [1.27].

Claim 2: now suppose that $\angle BGH + \angle DHG$ equals two right angles. Since $\angle AGH$ and $\angle BGH$ are adjacent angles, the sum $\angle AGH + \angle BGH$ equals two right angles by [1.13]. Or:

$$\begin{aligned} \angle BGH + \angle DHG &= \angle AGH + \angle BGH \\ \angle DHG &= \angle AGH \end{aligned}$$

which are alternate angles. By [1.27], $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; this proves our claim. □

PROPOSITION 1.29. *PARALLEL LINES IV.*

If a line intersects two parallel lines, then:

(1) corresponding alternate angles are equal to each other,

(2) exterior angles are equal to corresponding interior angles,

(3) the sum of interior angles on the same side equals two right angles.

PROOF. If $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ where $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, we claim that:

(1) alternate angles $\angle AGH$, $\angle GHD$ are equal;

(2) the exterior angle $\angle EGB$ equals its corresponding interior angle $\angle GHD$;

(3) the sum of the two interior angles, $\angle GHD + \angle HGB$, equals two right angles.

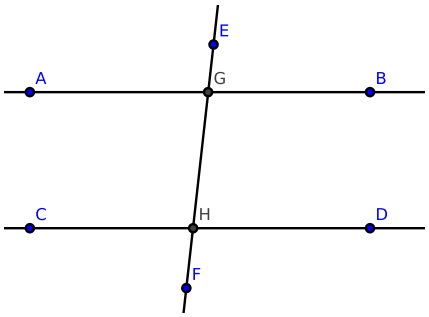


FIGURE 1.5.5. [1.29]

Claim 1: if $\angle AGH \neq \angle GHD$, one angle must be greater than the other. Wlog, suppose that $\angle AGH > \angle GHD$. Then we obtain the inequality

$$\angle AGH + \angle BGH > \angle GHD + \angle BGH$$

where $\angle AGH + \angle BGH$ is equal to the sum of two right angles by [1.13]. It follows that $\angle GHD + \angle BGH$ is less than two right angles.

By [Section 1.3.3, Axiom 4], $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ meet at some finite distance; this contradicts our hypothesis that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Therefore, $\angle AGH = \angle GHD$.

Claim 2: since $\angle EGB = \angle AGH$ by [1.15] and $\angle AGH = \angle GHD$ by Claim 1, it follows that $\angle EGB = \angle GHD$.

Claim 3. since $\angle AGH = \angle GHD$ by Claim 1, we have that

$$\angle AGH + \angle HGB = \angle GHD + \angle HGB$$

Since $\angle AGH + \angle HGB$ equals the sum of two right angle, $\angle GHD + \angle HGB$ equals the sum of two right angles. □

COROLLARY. *1.29.1. EQUIVALENT STATEMENTS REGARDING PARALLEL LINES. If two lines are intersected by a third line, then the two lines are parallel if and only if any one of these three properties hold:*

1) corresponding alternate angles are equal;

2) exterior angles equal their corresponding interior angles;

3) the sum of the interior angles on the same side are equal to two right angles.

COROLLARY. *1.29.2. We may replace the straight lines in [1.29, Cor. 1] with segments or rays, mutatis mutandis.*

Exercises.

Note: We may use [1.31] (that we may construct a straight line parallel to any given straight line) in the proofs of these exercises since the proof of [1.31] does not require [1.29].

1. Demonstrate both parts of [1.28] without using [1.27].

2. Construct $\overleftrightarrow{AB}$ containing the point C and $\overleftrightarrow{EF}$ containing the point D such that $\overleftrightarrow{AB} \parallel \overleftrightarrow{EF}$. Construct $\overline{CH}$ and $\overline{CJ}$ such that $\overline{CJ}$ bisects $\angle ACD$ and $\overline{CH}$ bisects $\angle BCD$. Prove that $\overline{DH} = \overline{DJ}$. [See the final chapter for a solution.]

4. If any other secant is constructed through the midpoint O of any line terminated by two parallel lines, the intercept on this line made by the parallels is bisected at O.

5. Two lines passing through a point which is equidistant from two parallel lines intercept equal segments on the parallels. [See the final chapter for a solution.]

6. Construct the perimeter of the parallelogram formed by constructing parallels to two sides of an equilateral triangle from any point in the third side. This perimeter is equal to $2\times$ the side.

7. If the opposite sides of a hexagon are equal and parallel, prove that its diagonals are concurrent.

8. If two intersecting segments are respectively parallel to two others, the angle between the former is equal to the angle between the latter. (Hint: if $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$ are respectively parallel to $\overleftrightarrow{DE}$, $\overleftrightarrow{DF}$ and if $\overleftrightarrow{AC}$, $\overleftrightarrow{DE}$ intersect at G, then the angles at points A, D are each equal to the angle at G [1.29].)

PROPOSITION 1.30. *TRANSITIVITY OF PARALLEL LINES.*

Lines parallel to the same line are also parallel to each other.

PROOF. Construct lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{EF}$ such that $\overleftrightarrow{AB} \parallel \overleftrightarrow{EF}$ and $\overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$. We claim that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

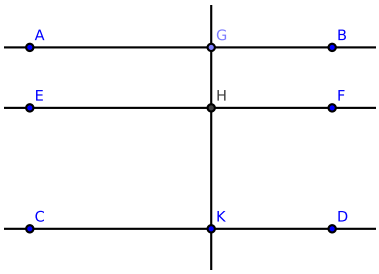


FIGURE 1.5.6. [1.30]

Construct any secant line $\overleftrightarrow{GHK}$. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{EF}$, the angle $\angle AGH = \angle GHF$ [Cor. 1.29.1]. Since $\overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$, the angle $\angle GHF = \angle HKD$ [Cor. 1.29.1]. It follows that $\angle AGH = \angle GHF = \angle HKD$. Since $\angle AGH = \angle AGK$ and $\angle HKD = \angle GKD$,

$$\angle AGK = \angle GKD$$

By [1.27], we have that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. □

COROLLARY. *1.30.1.* $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{EF}$, *and* $\overleftrightarrow{GK}$ *in [1.30] may be replaced by rays and/or line segments, mutatis mutandis.*

PROPOSITION 1.31. *CONSTRUCTION OF A PARALLEL LINE.*

We wish to construct a line which is parallel to a given line and passes through a given point.

PROOF. Given the line $\overleftrightarrow{AB}$ and a point C, we wish to construct the line $\overleftrightarrow{CE}$ such that $\overleftrightarrow{CE} \parallel \overleftrightarrow{AB}$.

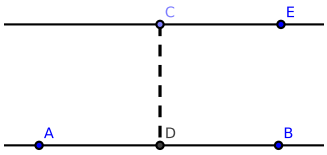


FIGURE 1.5.7. [1.31]

Take any point D on $\overleftrightarrow{AB}$ and construct $\overline{CD}$. Choose a point E such that $\angle DCE = \angle ADC$ [1.23], and construct $\overleftrightarrow{CE}$. By [Cor. 1.29.1], $\overleftrightarrow{CE} \parallel \overleftrightarrow{AB}$. □

COROLLARY. *1.31.1.* *$\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$ in [1.31] may be replaced by rays and/or line segments, mutatis mutandis.*

Definition:

43. The *altitude* of a triangle is the perpendicular segment from the triangle's base to the base's opposing vertex.

Exercises.

1. Given the altitude of a triangle and the base angles, construct the triangle. [See the final chapter for a solution.]

2. From a given point, construct a segment to a given segment such that the resultant angle is equal in measure to a given angle. Show that there are two solutions.

3. Prove the following construction for trisecting a given line $\overleftrightarrow{AB}$: on $\overleftrightarrow{AB}$, construct an equilateral $\triangle ABC$. Bisect the angles at points A and B by the lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BD}$. Through D, construct parallels to $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ which intersect $\overleftrightarrow{AB}$ at E and F. Claim: E and F are the points of trisection of $\overleftrightarrow{AB}$.

4. Inscribe a square in a given equilateral triangle such that its base stands on a given side of the triangle.

5. Through two given points on two parallel lines, construct two segments forming a rhombus with given parallels. [See the final chapter for a solution.]

6. Between two lines given in position, place a segment of given length which is parallel to a given line. Show that there are two solutions.

PROPOSITION 1.32. *EXTERIOR ANGLES AND SUMS OF ANGLES IN A TRIANGLE.*

By extending the side of one triangle, it can be shown that:

(1) the exterior angle equals the sum of the its interior and opposite angles;

(2) the sum of the three interior angles of the triangle equals two right angles.

PROOF. Construct $\triangle ABC$, and wlog extend side AB to segment $\overline{AD}$. We claim that:

(1) $\angle CBD = \angle BAC + \angle ACB$;

(2) $\angle BAC + \angle ACB + \angle ABC =$ two right angles.

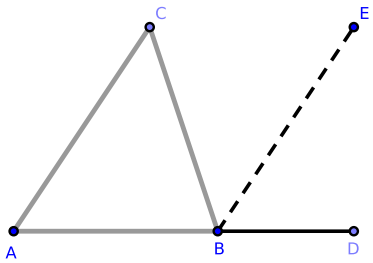


FIGURE 1.5.8. [1.32]

Claim 1: $\angle CBD = \angle BAC + \angle ACB$.

Construct $\overline{BE} \parallel \overline{AC}$ [1.31]. Since BC intersects the parallel segments $\overline{BE}$ and $\overline{AC}$, we have that $\angle EBC = \angle ACB$ [1.29]. Also, since AB intersects the parallel segments $\overline{BE}$ and $\overline{AC}$, we have that $\angle DBE = \angle BAC$ [1.29]. Since $\angle CBD = \angle EBC + \angle DBE$, we have

$$\angle CBD = \angle ACB + \angle BAC$$

Claim 2: $\angle BAC + \angle ACB + \angle ABC =$ two right angles.

Adding $\angle ABC$ to each side of the equality in Claim 1, we obtain

$$\angle ABC + \angle CBD \quad = \quad \angle ABC + \angle ACB + \angle BAC$$

Since $\angle ABC + \angle CBD$ equals two right angles by [1.13], the sum $\angle BAC + \angle ACB + \angle ABC$ equals two right angles. This proves our claim. □

COROLLARY. *1.32.1. If a right triangle is isosceles, then each base angle equals half of a right angle.*

COROLLARY. *1.32.2. If two triangles have two angles in one respectively equal to two angles in the other, then their remaining pair of angles is also equal.*

COROLLARY. *1.32.3. Since a quadrilateral can be divided into two triangles, the sum of its angles equals four right angles.*

COROLLARY. *1.32.4. If a figure of n sides is divided into triangles by drawing diagonals from any one of its angles, we will obtain $(n-2)$ triangles. Hence, the sum of its angles equals $2(n-2)$ right angles.*

COROLLARY. *1.32.5. If all the sides of any convex polygon are extended, then the sum of the external angles equals to four right angles.*

COROLLARY. *1.32.6. Each angle of an equilateral triangle equals two-thirds of a right angle.*

COROLLARY. *1.32.7. If one angle of a triangle equals the sum of the other two, then it is a right angle.*

COROLLARY. *1.32.8. Every right triangle can be divided into two isosceles triangles by a line constructed from the right angle to the hypotenuse.*

Exercises.

1. Trisect a right angle.

2. If the sides of a polygon of n sides are extended, then the sum of the angles between each alternate pair is equal to $2(n-4)$ right angles.

3. If the line which bisects an external vertical angle of a triangle is parallel to the base of the triangle, then the triangle is isosceles. [See the final chapter for a solution.]

4. If two right triangles $\triangle ABC$, $\triangle ABD$ are on the same hypotenuse AB and if the vertices C and D are joined, then the pair of angles standing opposite any side of the resulting quadrilateral are equal.

5. Prove that the three altitudes of a triangle are concurrent. Note: We are proving the existence of the **orthocenter**[25] of a triangle: the point where the three altitudes intersect, and one of a triangle's **points of concurrency**[26]. [See the final chapter for a solution.]

6. The bisectors of two adjacent angles of a parallelogram stand at right angles. [See the final chapter for a solution.]

7. The bisectors of the external angles of a quadrilateral form a circumscribed quadrilateral, the sum of whose opposite angles equals two right angles.

8. If the three sides of one triangle are respectively perpendicular to those of another triangle, the triangles are equiangular. (This problem may be delayed until the end of chapter 1.)

9. Construct a right triangle being given the hypotenuse and the sum or difference of the sides.

10. The angles made with the base of an isosceles triangle by altitudes from its endpoints on the equal sides are each equal to half the vertical angle.

11. The angle included between the internal bisector of one base angle of a triangle and the external bisector of the other base angle is equal to half the vertical angle.

12. In the construction of [1.18], prove that the angle $\angle DBC$ is equal to half the difference of the base angles.

13. If A, B, C denote the angles of a triangle, prove that $\frac{1}{2}(A+B)$, $\frac{1}{2}(B+C)$, and $\frac{1}{2}(A+C)$ are the angles of a triangle formed by any side, the bisectors of the external angles between that side, and the other extended sides.

14. Prove [Cor. 1.32.1].

15. Prove [Cor. 1.32.2].

16. Prove [Cor. 1.32.3].

17. Prove [Cor. 1.32.4].

18. Prove [Cor. 1.32.5].

19. Prove [Cor. 1.32.6].

20. Prove [Cor. 1.32.7].

21. Prove [Cor. 1.32.8].

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[25] http://mathworld.wolfram.com/Orthocenter.html

[26] http://www.mathopenref.com/concurrentpoints.html

PROPOSITION 1.33. *PARALLEL SEGMENTS.*

Segments which join adjacent endpoints of two equal, parallel segments are themselves parallel and equal in length.

PROOF. Suppose that $\overline{AB} \parallel \overline{CD}$ and $\overline{AB} = \overline{CD}$. Construct $\boxdot ABDC$. We claim that:

(1) $\overline{AC} = \overline{BD}$;

(2) $\overline{AC} \parallel \overline{BD}$.

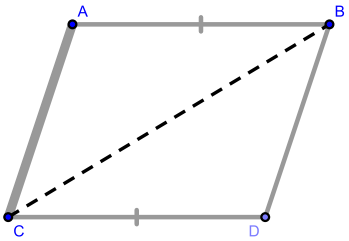

FIGURE 1.5.9. [1.33]

Construct $\overline{BC}$. Since $\overline{AB} \parallel \overline{CD}$ by hypothesis and $\overline{BC}$ intersects them, we have that $\angle ABC = \angle DCB$ [1.29].

Consider $\triangle ABC$ and $\triangle DCB$: $AB = DC$, the triangles share side BC, and $\angle ABC = \angle DCB$. By [1.4], $\triangle ABC \cong \triangle DCB$, and so $\overline{AC} = \overline{BD}$ (claim 1).

Also by [1.4], $\angle ACB = \angle CBD$. By [1.27], $\overline{AC} \parallel \overline{BD}$ (claim 2). This proves our claim. □

COROLLARY. *1.33.1. [1.33] holds for straight lines and rays, mutatis mutandis.*

COROLLARY. *1.33.2. Figure $\boxdot ABDC$ is a parallelogram [Def. 1.39].*

Exercises.

1. Prove that if two segments $\overline{AB}$, $\overline{BC}$ are respectively equal and parallel to two other segments $\overline{DE}$, $\overline{EF}$, then the segment $\overline{AC}$ joining the endpoints of the former pair is equal in length to the segment $\overline{DF}$ joining the endpoints of the latter pair. [See the final chapter for a solution.]

PROPOSITION 1.34. *OPPOSITE SIDES AND OPPOSITE ANGLES OF PARALLELOGRAMS.*

The opposite sides and the opposite angles of a parallelogram are equal to one another and either diagonal bisects the parallelogram.

PROOF. Construct $\boxdot ABCD$. We claim that:

(1)$\overline{AB} = \overline{CD}$ and $\overline{AC} = \overline{BD}$;

(2)$\angle CAB = \angle CDB$;

(3)$\angle ACD = \angle ABD$;

(4) either diagonal ($\overline{BC}$ or $\overline{AD}$) bisects the parallelogram.

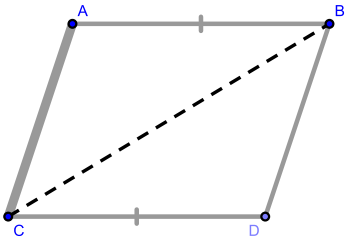

FIGURE 1.5.10. [1.34]

Construct $\overline{BC}$, and consider $\triangle ABC$ and $\triangle DBC$: since $\overline{AB} \parallel \overline{CD}$ by construction and $\overline{BC}$ intersects them, $\angle ABC = \angle DCB$ and $\angle ACB = \angle CBD$ [1.29]. Also, the triangles share side $\overline{BC}$. By [1.26], $\triangle ABC \cong \triangle DCB$: it follows that $\overline{AB} = \overline{CD}$ and $\overline{AC} = \overline{BD}$ (claim 1) as well as $\angle CAB = \angle CDB$ (claim 2).

Now $\angle ACD = \angle ACB + \angle DCB$ and $\angle ABD = \angle CBD + \angle ABC$. Since $\angle ACB = \angle CBD$ and $\angle DCB = \angle ABC$ by [Cor. 1.29.1], we obtain

$$\begin{aligned} \angle ACD &= \angle ACB + \angle DCB \\ &= \angle CBD + \angle ABC \\ &= \angle ABD \end{aligned}$$

(claim 3).

Finally, since $\boxdot ABCD = \triangle ABC \oplus \triangle DEF$ and $\triangle ABC \cong \triangle DCB$ (and hence the triangles have that same area), the diagonal bisects the parallelogram. The remaining case follows *mutatis mutandis* if we construct AD instead of BC (claim 4). □

COROLLARY. *1.34.1. The area of $\boxdot ABDC$ is double the area of either $\triangle ACB$ or $\triangle BCD$.*

COROLLARY. *1.34.2. If one angle of a parallelogram is a right angle, each of its angles are right angles.*

COROLLARY. *1.34.3. If two adjacent sides of a parallelogram are equal in length, then it is a lozenge.*

COROLLARY. *1.34.4. If both pairs of opposite sides of a quadrilateral are equal in length, it is a parallelogram.*

COROLLARY. *1.34.5. If both pairs of opposite angles of a quadrilateral are equal, it is a parallelogram.*

COROLLARY. *1.34.6. If the diagonals of a quadrilateral bisect each other, it is a parallelogram.*

COROLLARY. *1.34.7. If both diagonals of a quadrilateral bisect the quadrilateral, it is a parallelogram.*

COROLLARY. *1.34.8. If the adjacent sides of a parallelogram are equal, its diagonals bisect its angles.*

COROLLARY. *1.34.9. If the adjacent sides of a parallelogram are equal, its diagonals intersect at right angles.*

COROLLARY. *1.34.10. In a right parallelogram, the diagonals are equal in length.*

COROLLARY. *1.34.11. If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a rhombus.*

COROLLARY. *1.34.12. If a diagonal of a parallelogram bisects the angles whose vertices it joins, the parallelogram is a rhombus.*

Exercises.

1. Prove that the diagonals of a parallelogram bisect each other. [See the final chapter for a solution.]

2. If the diagonals of a parallelogram are equal, then each of its angles are right angles. [See the final chapter for a solution.]

3. The segments joining the adjacent endpoints of two unequal parallel segments will meet when extended on the side of the shorter parallel.

4. If two opposite sides of a quadrilateral are parallel but unequal in length and the other pair are equal but not parallel, then its opposite angles are supplemental.

5. Construct a triangle after being given the midpoints of its three sides.

6. Prove [Cor. 1.34.1].

7. Prove [Cor. 1.34.2].

8. Prove [Cor. 1.34.3].

9. Prove [Cor. 1.34.4].

10. Prove [Cor. 1.34.5].

11. Prove [Cor. 1.34.6].

12. Prove [Cor. 1.34.7].

13. Prove [Cor. 1.34.8].

14. Prove [Cor. 1.34.9].

15. Prove [Cor. 1.34.10].

16. Prove [Cor. 1.34.11].

17. Prove [Cor. 1.34.12].

PROPOSITION 1.35. *AREAS OF PARALLELOGRAMS I.*

Parallelograms on the same base and between the same parallels are equal in area.

PROOF. We claim that parallelograms constructed on base $\overline{BC}$ and between parallels $\overline{AF}$ and $\overline{BC}$ are equal in area.

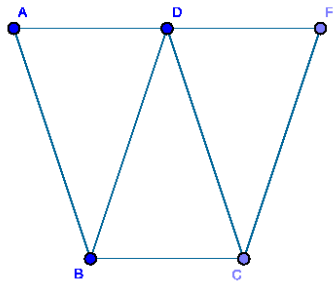

FIGURE 1.5.11. [1.35], case 1

Case 1: construct the parallelograms $\boxdot ADCB$, $\boxdot FDBC$ on the common base BC as in Fig. 1.5.11. By [Cor. 1.34.1],

$$\boxdot ADCB = 2 \cdot \triangle BCD = \boxdot FDBC$$

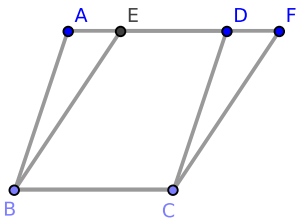

FIGURE 1.5.12. [1.35], case 2

Case 2: construct the parallelograms $\boxdot ADCB$ and $\boxdot EFCB$ as in Fig. 1.5.12. Because $\boxdot ABCD$ is a parallelogram, $AD = BC$ [1.34]; because $\boxdot BCEF$ is a parallelogram, $EF = BC$. Hence, $AD = EF$. It follows that

$$\begin{aligned} \overline{AD} - \overline{ED} &= \overline{EF} - \overline{ED} \\ \overline{AE} &= \overline{DF} \end{aligned}$$

Consider $\triangle BAE$ and $\triangle CDF$: $BA = CD$, $AE = DF$, and $\angle BAE = \angle CDF$ by [1.29, Cor. 1]. By [1.4], $\triangle BAE = \triangle CDF$. Notice that

$$\begin{aligned} AFCB &= \boxdot EFCB + \triangle BAE \\ &= \boxdot ADCB + \triangle CDF \end{aligned}$$

Since $\triangle BAE = \triangle CDF$, we have that $\boxdot EFCB = \boxdot ADCB$.

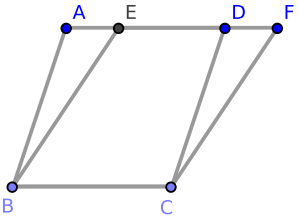

FIGURE 1.5.13. [1.35], case 2

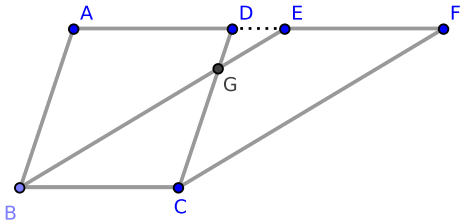

FIGURE 1.5.14. [1.35], case 3

Case 3: construct the parallelograms $\boxdot ADCB$ and $\boxdot EFCB$ as in Fig. 1.5.13. As in case 2, $AD = BC = EF$. Construct $\overline{DE}$.

Consider $\triangle BAE$ and $\triangle CDF$: $BA = CD$, $AE = DF$, and $\angle BAE = \angle CDF$ by [1.29]. By [1.4], $\triangle BAE = \triangle CDF$, from which it follows:

$$\begin{aligned} \triangle BAE - \triangle DEG &= \triangle CDF - \triangle DEG \\ ADGB &= CGEF \\ ADGB + \triangle BGC &= CGEF + \triangle BGC \\ \boxdot ADCB &= \boxdot EFCB \end{aligned}$$

This proves our claim. □

PROPOSITION 1.36. *AREAS OF PARALLELOGRAMS II.*

Parallelograms on equal bases and on the same parallels are equal in area.

PROOF. Let $\boxdot ADCB$ and $\boxdot EHGF$ be constructed on $\overline{BC}$ and $\overline{FG}$ where $\overline{BC} = \overline{FG}$ and between $\overleftrightarrow{AH} \parallel \overleftrightarrow{BG}$. We claim that $\boxdot ADCB = \boxdot EHGF$.

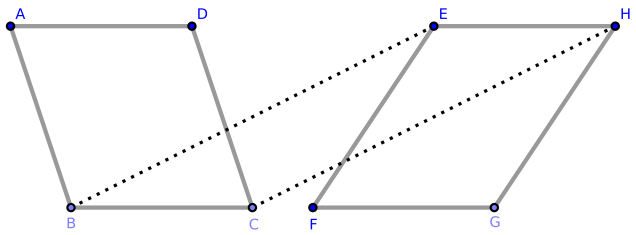

FIGURE 1.5.15. [1.36]

Construct $\overline{BE}$ and $\overline{CH}$. Since $\boxdot EHGF$ is a parallelogram, $\overline{FG} = \overline{EH}$ [1.34]. Since $\overline{BC} = \overline{FG}$ by hypothesis, $\overline{BC} = \overline{EH}$. Given this equality and $\overline{BC} \parallel \overline{EH}$, by [1.33] $\overline{BE} = \overline{CH}$ and $\overline{BE} \parallel \overline{CH}$. It follows that $\boxdot EHCB$ is a parallelogram.

By [1.35], $\boxdot EHCB = \boxdot EHGF$ and $\boxdot EHCB = \boxdot ADCB$. Therefore, $\boxdot ADCB = \boxdot EHGF$, which proves our claim. □

PROPOSITION 1.37. *TRIANGLES OF EQUAL AREA I.*

Triangles which stand on the same base and in the same parallels are equal in area.

PROOF. Construct $\triangle ABC$ and $\triangle DBC$ on base $\overline{BC}$ such that each triangle stands between parallels $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$. We claim that $\triangle ABC = \triangle DBC$.

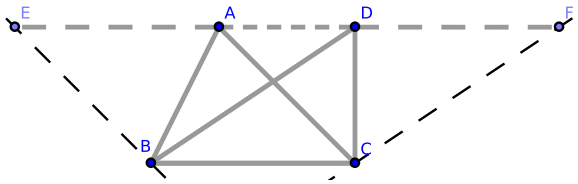

FIGURE 1.5.16. [1.37]

Construct $\overleftrightarrow{BE} \parallel \overleftrightarrow{AC}$ and $\overleftrightarrow{CF} \parallel \overleftrightarrow{BD}$. It follows that $\boxdot AEBC$ and $\boxdot DBCF$ are parallelograms. By [1.35], $\boxdot AEBC = \boxdot DBCF$.

Notice that $\frac{1}{2} \cdot \triangle ABC = \boxdot AEBC$ because the diagonal AB bisects $\boxdot AEBC$ [1.34, #1]. Similarly, $\frac{1}{2} \cdot \triangle DBC = \boxdot DBCF$. It follows that

$$\begin{aligned} \frac{1}{2} \cdot \triangle ABC &= \frac{1}{2} \cdot \triangle DBC \\ \triangle ABC &= \triangle DBC \end{aligned}$$

□

Exercises.

1. If two triangles of equal area stand on the same base but on opposite sides of the base, the segment connecting their vertices is bisected by the base or its extension. [See the final chapter for a solution.]

2. Construct a triangle equal in area to a given quadrilateral figure.

3. Construct a triangle equal in area to a given polygon.

4. Construct a rhombus equal in area to a given parallelogram and having a given side of the parallelogram as the base.

PROPOSITION 1.38. *TRIANGLES OF EQUAL AREA II.*

Triangles which stand on equal bases and in the same parallels are equal in area.

PROOF. Construct $\triangle ABC$ and $\triangle DEF$ between $\overleftrightarrow{BF}$ and $\overleftrightarrow{AD}$ such that $\overleftrightarrow{BF} \parallel \overleftrightarrow{AD}$ and $BC = EF$. We claim that $\triangle ABC = \triangle DEF$.

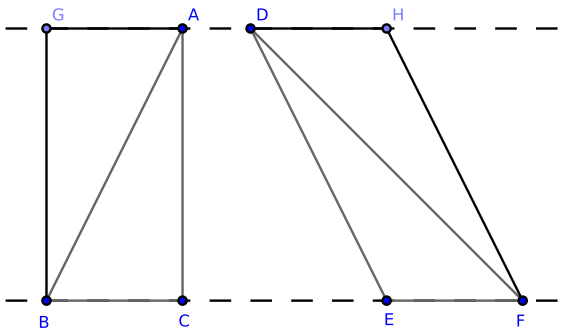

FIGURE 1.5.17. [1.38]

By [1.31], construct $\overline{BG}$ where G is a point on $\overleftrightarrow{AD}$ and where $\overline{BG} \parallel \overline{AC}$; similarly, construct $\overline{FH}$ where H is a point on $\overleftrightarrow{AD}$ and where $\overline{FH} \parallel \overline{DE}$.

It follows that $\boxdot GACB$ and $\boxdot DHFE$ are parallelograms. Since $BC = EF$, by [1.36] $\boxdot GACB = \boxdot DHFE$.

Since $\overline{AB}$ bisects $\boxdot GACB$ and $\overline{DF}$ bisects $\boxdot DHFE$ by [1.34], $\triangle ABC = \triangle DEF$. This proves our claim. □

Exercises.

1. Every median of a triangle bisects the triangle. [See the final chapter for a solution.]

2. If two triangles have two sides of one respectively equal to two sides of the other and where the interior angles are supplemental, then their areas are equal.

3. If the base of a triangle is divided into any number of equal segments, then segments constructed from the vertex to the points of division divide the whole triangle into as many equal parts.

4. The diagonal of a parallelogram and segments from any point on the diagonal to the vertices through which the diagonal does not pass divide the parallelogram into four triangles which are equal (in a two-by-two fashion).

5. One diagonal of a quadrilateral bisects the other if and only if the diagonal also bisects the quadrilateral. [See the final chapter for a solution.]

6. If $\triangle ABC$ and $\triangle ABD$ each stand on the base AB and between the same parallels, and if a parallel to AB meets the sides AC and BC at the points E and F and meets the sides AD and BD at the points G, H, then $\overline{EF} = \overline{GH}$.

7. If instead of triangles on the same base we have triangles on equal bases and between the same parallels, the intercepts made by the sides of the triangles on any parallel to the bases are equal in length.

8. If the midpoints of any two sides of a triangle are joined, the triangle formed with the two half sides has an area equal to one-fourth of the whole.

9. The triangle whose vertices are the midpoints of two sides and any point in the base of another triangle has an area equal to one-fourth the area of that triangle.

10. Bisect a given triangle by a segment constructed from a given point in one of the sides.

11. Trisect a given triangle by three segments constructed from a given point within it.

12. Prove that any segment through the intersection of the diagonals of a parallelogram bisects the parallelogram.

13. The triangle formed by joining the midpoint of one of the non-parallel sides of a trapezoid to the endpoints of the opposite side is equal in area to half the area of the trapezoid. (Recall that a trapezoid is any rectilinear quadrilateral plane figure which is not a parallelogram.)

PROPOSITION 1.39. *TRIANGLES OF EQUAL AREA III.*

Triangles which are equal in area and stand on the same base and on the same side of the base also stand between the same parallels.

PROOF. Suppose that $\triangle BAC$ and $\triangle BDC$ stand on the same base (BC), on the same side of BC, and that $\triangle BAC = \triangle BDC$; we claim $\triangle BAC$ and $\triangle BDC$ also stand between the same parallels ($\overline{AD}$, $\overline{BC}$).

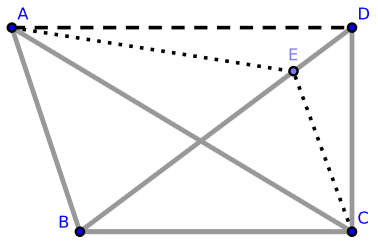

FIGURE 1.5.18. [1.39]

Construct $\overline{AD}$. Clearly, the triangles stand between $\overline{AD}$ and $\overline{BC}$. We need only prove that $\overline{AD} \parallel \overline{BC}$.

We shall prove our claim with a proof by contradiction. If $\overline{AD} \nparallel \overline{BC}$, suppose that $\overline{AE} \parallel \overline{BC}$ where E is a point on $\overline{BD}$; construct $\overline{EC}$. Notice that $\triangle EDC > 0$.

Since the triangles $\triangle BEC$, $\triangle BAC$ stand on the same base BC and between the same parallels ($\overline{BC}$ and $\overline{AE}$), we have that $\triangle BEC = \triangle BAC$ [1.37]. By hypothesis, $\triangle BAC = \triangle BDC$. Therefore, $\triangle BEC = \triangle BDC$ [Axiom 1.1]. But $\triangle BDC = \triangle BEC + \triangle EDC$, so $\triangle EDC = 0$, a contradiction. This proves our claim that $\overline{AD} \parallel \overline{BC}$. □

PROPOSITION 1.40. *TRIANGLES OF EQUAL AREA IV.*

Triangles which are equal in area and stand on equal bases on the same side of their bases stand between the same parallels.

PROOF. Construct $\triangle ABC$ and $\triangle DEF$ on $\overline{BCEF}$ such that $BC = EF$, $\overline{CF} > 0$, $\triangle ABC = \triangle DEF$, and where each triangle stands on the same side of its base. Construct $\overline{AD}$. Clearly, $\triangle ABC$ and $\triangle DEF$ stand between $\overline{AD}$ and $\overline{BF}$; we claim that $\overline{AD} \parallel \overline{BF}$.

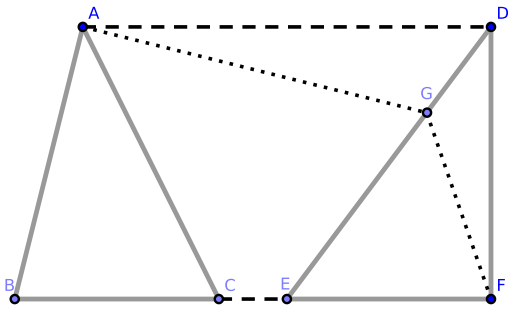

FIGURE 1.5.19. [1.40]

If $\overline{AD} \nparallel \overline{BF}$, construct $\overline{AG}$ (where G is a point on $\overline{DE}$) such that $\overline{AG} \parallel \overline{BF}$. Also construct $\overline{FG}$. Notice that $\triangle DFG > 0$.

Consider the triangles $\triangle GEF$ and $\triangle ABC$: they stand on equal bases (BC, EF) and between the same parallels ($\overline{BF}$, $\overline{AG}$), and so by [1.38] $\triangle GEF = \triangle ABC$.

But $\triangle DEF = \triangle ABC$ by hypothesis, and so $\triangle DEF = \triangle GEF$. Since $\triangle DEF = \triangle GEF + \triangle DGF$, $\triangle DGF = 0$, a contradiction. Therefore, $\overline{AD} \parallel \overline{BF}$. □

Exercises.

1. Prove that triangles with equal bases and altitudes are equal in area. [See the final chapter for a solution.]

2. The segment joining the midpoints of two sides of a triangle is parallel to the base, and the medians from the endpoints of the base to these midpoints will each bisect the original triangle. Hence, the two triangles whose base is the third side and whose vertices are the points of bisection are equal in area. [See the final chapter for a solution.]

3. The parallel to any side of a triangle through the midpoint of another bisects the third.

4. The segments which connect the midpoints of the sides of a triangle divide the triangle into four congruent triangles. [See the final chapter for a solution.]

5. The segment which connects the midpoints of two sides of a triangle is equal in length to half the third side.

6. The midpoints of the four sides of a convex quadrilateral, taken in order, are the vertices of a parallelogram whose area is equal to half the area of the quadrilateral.

7. The sum of the two parallel sides of a trapezoid is double the length of the segment joining the midpoints of the two remaining sides.

8. The parallelogram formed by the segment which connects the midpoints of two sides of a triangle and any pair of parallels constructed through the same points to meet the third side is equal in area to half the area of the triangle.

9. The segment joining the midpoints of opposite sides of a quadrilateral and the segment joining the midpoints of its diagonals are concurrent.

PROPOSITION 1.41. *PARALLELOGRAMS AND TRIANGLES.*

If a parallelogram and a triangle stand on the same base and between the same parallels, then the parallelogram is double the area of the triangle.

PROOF. Suppose a parallelogram ($\boxdot ABCD$) and a triangle ($\triangle EBC$) stand on the same base (BC) and between the same parallels ($\overline{AE}$, $\overline{BC}$); we claim that the area of the parallelogram is double the area of the triangle.

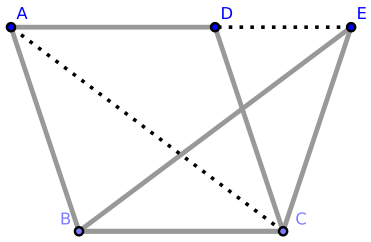

FIGURE 1.5.20. [1.41]

Construct the above and $\overline{AC}$ and $\overline{DE}$. Clearly, $\triangle EBC$ stands on the same base and between the same parallels as $\boxdot ABCD$.

By [1.34], $\boxdot ABCD = 2 \cdot \triangle ABC$; by [1.37], $\triangle ABC = \triangle EBC$. Therefore, $\boxdot ABCD = 2 \cdot \triangle EBC$, which proves our claim. □

COROLLARY. *1.41.1. If a triangle and a parallelogram have equal altitudes and if the base of the triangle is double of the base of the parallelogram, then their areas are equal.*

COROLLARY. *1.41.2. Suppose we have two triangles whose bases are two opposite sides of a parallelogram and which have any point between these sides as a common vertex. Then the sum of the areas of these triangles equals half the area of the parallelogram.*

PROPOSITION 1.42. *CONSTRUCTION OF PARALLELOGRAMS I.*

Given an arbitrary triangle and an arbitrary acute angle, it is possible to construct a parallelogram equal in area to the triangle which contains the given angle.

PROOF. Construct $\triangle ABC$ and $\angle RDS$. We wish to construct a parallelogram which:

(1) equals the area of $\triangle ABC$;

(2) contains an angle equal to $\angle RDS$.

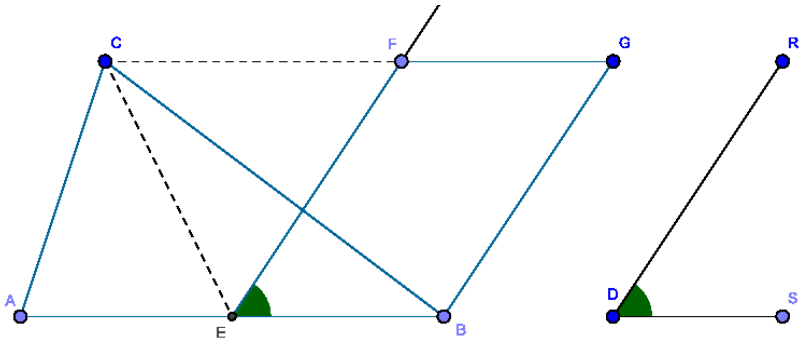

FIGURE 1.5.21. [1.42]

Bisect AB at E, and construct $\overline{EC}$. Construct $\angle BEF = \angle RDS$ [1.23], $\overline{CG} \parallel \overline{AB}$, and $\overline{BG} \parallel \overline{EF}$ [1.31]. We claim that $\boxdot FGBE$ is the required parallelogram.

Because $\overline{AE} = \overline{EB}$ by construction, $\triangle AEC = \triangle EBC$ by [1.38]. Therefore, $\triangle ABC = 2 \cdot \triangle EBC$. We also have that $\boxdot FGBE = 2 \cdot \triangle EBC$ because each stands on the same base (EB) and between the same parallels (EB and CG) [1.41]. Therefore, $\boxdot FGBE = \triangle ABC$ (claim 1), and by construction, $\boxdot FGBE$ contains $\angle BEF$ where $\angle BEF = \angle RDS$ (claim 2). $\square$

PROPOSITION 1.43. *COMPLEMENTARY AREAS OF PARALLELOGRAMS.*

Parallel segments through any point in one of the diagonals of a parallelogram divides the parallelogram into four smaller parallelograms: the two through which the diagonal does not pass are called the *complements* of the other two parallelograms, and these complements are equal in area.

PROOF. Construct $\boxdot ADCB$ and diagonal $\overline{AC}$. Let K be any point on $\overline{AC}$ except A or C. Construct $\overline{GH}$ and $\overline{EF}$ through K such that $\overline{GH} \parallel \overline{CD}$ and $\overline{EF} \parallel \overline{AD}$. This divides $\boxdot ADCB$ into four smaller parallelograms: the two through which the diagonal does not pass ($\boxdot EKGB$, $\boxdot HDFK$) are called the *complements* of the two parallelograms through which the diagonal passes. We claim that $\boxdot EBGK = \boxdot HKFD$.

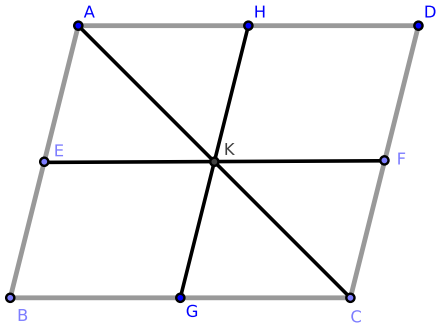

FIGURE 1.5.22. [1.43]

Because $\overline{AC}$ bisects the parallelograms $\boxdot ADCB$, $\boxdot AHKE$, and $\boxdot KFCG$, we have that $\triangle ADC = \triangle ABC$, $\triangle AHK = \triangle AEK$, and $\triangle KFC = \triangle KGC$ [1.34]. Therefore,

$$\begin{aligned} \boxdot EKGB &= \triangle ABC - \triangle AEK - \triangle KGC \\ &= \triangle ADC - \triangle AHK - \triangle KFC \\ &= \boxdot HDFK \end{aligned}$$

which proves our claim. □

COROLLARY. *1.43.1. If through some point K within parallelogram $\boxdot ABCD$ we have constructed parallel segments to its sides in order to make the parallelograms $\boxdot HDFK$, $\boxdot EKGB$ equal in area, then K is a point on the diagonal $\overline{AC}$.*

Therefore, $\boxdot HDFK$, $\boxdot EKGB$ equal in area if and only if K is a point on the diagonal $\overline{AC}$.

COROLLARY. *1.43.2.* *$\boxdot AHGB = \boxdot ADFE$ and $\boxdot EFCB = \boxdot HDCG$.*

Exercises.

1. Prove [1.43, Cor. 1].
2. Prove [1.43, Cor. 2].

PROPOSITION 1.44. *CONSTRUCTION OF PARALLELOGRAMS II.*

Given an arbitrary triangle, an arbitrary angle, and an arbitrary segment, we can construct a parallelogram equal in area to the triangle which contains the given angle and has a side length equal to the given segment.

PROOF. Construct $\overline{AB}$; we wish to construct a parallelogram which:

(1) is equal in area to $\triangle NPQ$

(2) contains an angle equal to $\angle RST$

(3) is constructed on $\overline{AB}$.

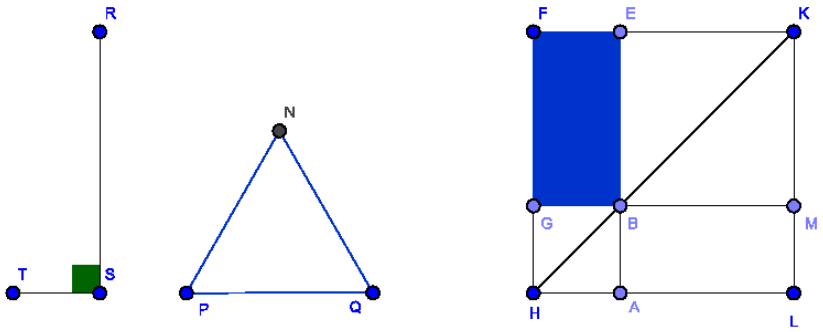

FIGURE 1.5.23. [1.44]

Construct the parallelogram $\boxdot BEFG$ where $\boxdot BEFG = \triangle NPQ$ [1.42], $\angle GBE = \angle RST$, and $\overline{AB} \oplus \overline{BE} = \overline{AE}$. Also construct segment $\overline{AH} \parallel \overline{BG}$ [1.31]. Extend $\overline{FG}$ to $\overline{FH}$, and construct $\overline{HB}$.

Since $\overline{AH} \parallel \overline{BG}$ and $\overline{BG} \parallel \overline{FE}$ by construction, $\overline{AH} \parallel \overline{FE}$ by [1.30]. Notice that $\overline{HF}$ intersects $\overline{AH}$ and $\overline{FE}$; therefore, $\angle AHF + \angle HFE =$ two right angles. It follows that $\angle BHG + \angle GFE <$ two right angles. By [section 1.3.3, Axiom 4], if we extend $\overline{HB}$ and $\overline{FE}$, they will intersect at some point K. Through K, construct $\overline{KL} \parallel \overline{AB}$ [1.31] and extend $\overline{AH}$ to intersect $\overline{KL}$ at L, and extend $\overline{GB}$ to intersect $\overline{KL}$ at M. We claim that $\boxdot BMLA$ fulfills the required conditions.

By [1.43], $\boxdot BMLA = \boxdot FEBG$, and $\boxdot FEBG = \triangle NPQ$ by construction; therefore $\boxdot BALM = \triangle NPQ$ (claim 1).

By [1.15], $\angle ABM = \angle EBG$, and $\angle EBG = \angle RST$ by construction; therefore, $\angle ABM = \angle RST$ (claim 2).

Finally, $\boxdot BALM$ is constructed on $\overline{AB}$ (claim 3). □

PROPOSITION 1.45. *CONSTRUCTION OF PARALLELOGRAMS III.*

Given an arbitrary angle and an arbitrary polygon, it is possible to construct a parallelogram equal in area to the polygon which contains an angle equal to the given angle.

PROOF. Construct polygon $ABCD$ and $\angle LMN$. We wish to construct a parallelogram which:

(1) contains an angle equal to a $\angle LMN$;

(2) equals the area of $ABCD$.

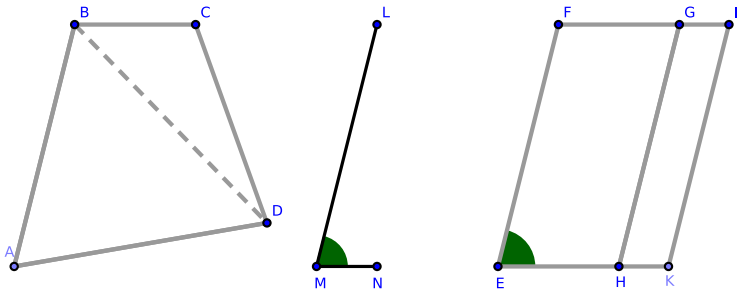

FIGURE 1.5.24. [1.45]

By [1.42]. we may construct $\overline{BD}$ and $\boxdot FGHE$ such that $\boxdot FGHE = \triangle ABD$ where $\angle FEH = \angle LMN$.

On $\overline{GH}$, construct $\boxdot GIKH$ such that $\boxdot GIKH = \triangle BCD$ where $\angle GHK = \angle LMN$ [1.44]. We may continue to this algorithm for any additional triangles that remain in $ABCD$. Upon completing this algorithm, we claim that $\boxdot FEKI$ fulfills the required conditions.

Because $\angle GHK = \angle LMN = \angle FEH$ by construction, $\angle GHK = \angle FEH$. From this, we obtain

$$\angle GHK + \angle GHE \quad = \quad \angle FEH + \angle GHE$$

Since $\overline{HG} \parallel \overline{EF}$ and $\overline{EH}$ intersects them, the sum $\angle FEH + \angle GHE$ equals two right angles [1.29]. Hence, $\angle GHK + \angle GHE$ equals two right angles, and so $\overline{EH} \oplus \overline{HK} = \overline{EK}$ [1.14, Cor. 1].

Similarly, because $\overline{GH}$ intersects the parallels $\overline{FG}$ and $\overline{EK}$, $\angle FGH = \angle GHK$ [1.29]. From this, we obtain

$$\angle FGH + \angle HGI \quad = \quad \angle GHK + \angle HGI$$

Since $\overline{GI} \parallel \overline{HK}$ and $\overline{GH}$ intersects them, the sum $\angle GHK + \angle HGI$= two right angles [1.29]. Hence, $\angle FGH + \angle HGI$= two right angles, and $\overline{FG} \oplus \overline{GI} = \overline{FI}$.

Because $\boxdot FEHG$ and $\boxdot GHKI$ are parallelograms, $\overline{EF}$ and $\overline{KI}$ are each parallel to $\overline{GH}$; by [1.30], $\overline{EF} \parallel \overline{KI}$ and $\overline{EK} \parallel \overline{FI}$. Therefore, $\boxdot FIKE$ is a parallelogram.

Since $\angle FEK = \angle FEH = \angle LMN$, $\boxdot FIKE$ contains an angle equal to a $\angle LMN$ (claim 1).

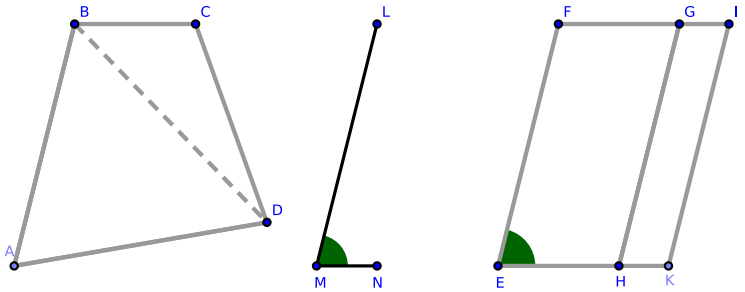

FIGURE 1.5.25. [1.45]

And because the parallelogram $\boxdot FGHE = \triangle ABD$ by construction and $\boxdot GIKH = \triangle BCD$,

$$\begin{aligned} ABCD &= \triangle ABD \oplus \triangle BCD \\ &= \boxdot FGHE \oplus \boxdot GIKH \\ &= \boxdot FIKE \end{aligned}$$

(claim 2). □

Exercises.

1. Construct a rectangle equal to the sum of $2, 3, ..., n$ number of polygons.

2. Construct a rectangle equal in area to the difference in areas of two given figures.

PROPOSITION 1.46. *CONSTRUCTION OF A SQUARE I.*

Given an arbitrary segment, we may construct a square on that segment.

PROOF. Construct $\overline{AB}$; we wish to construct a square on $\overline{AB}$.

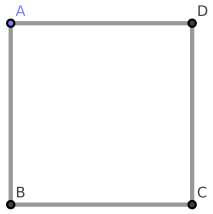

FIGURE 1.5.26. [1.46]

Construct $\overline{AD} \perp \overline{AB}$ [1.11] where $\overline{AD} = \overline{AB}$ [1.3]. Through D, construct $\overline{CD} \parallel \overline{AB}$ [1.31] where $\overline{AB} = \overline{CD}$, and through B construct $\overline{BC} \parallel \overline{AD}$. We claim that $\boxdot ABCD$ is the required square.

By construction,

$$\overline{AD} = \overline{AB} = \overline{CD}$$

Because $\boxdot ABCD$ is a parallelogram, $\overline{AD} = \overline{BC}$ [1.34]; hence, all four sides of $\boxdot ABCD$ are equal. It follows that $\boxdot ABCD$ is a rhombus and $\angle DAB$ is a right angle. By [Def. 1.30], $\boxdot ABCD$ is a square. □

REMARK. See also [2.14].

Exercises.

1. Prove that two squares have equal side-lengths if and only if the squares are equal in area. [See the final chapter for a solution.]

2. The parallelograms about the diagonal of a square are squares.

3. If on the four sides of a square (or on the sides which are extended) points are taken which are equidistant from the four angles, then they will be the vertices of another square (and similarly for a regular pentagon, hexagon, etc.).

4. Divide a given square into five equal parts: specifically, four right triangles and a square.

5. Prove that the formula for the area of a rectangle is $A = bh$.

PROPOSITION 1.47. *THE PYTHAGOREAN THEOREM (aka. THE GOUGU THEOREM).*

In a right triangle, the square on the side opposite the right angle (the hypotenuse) is equal in area to the sum of the areas of the squares on the remaining sides.

PROOF. Construct right triangle $\triangle ABC$ where AB is the hypotenuse. We claim that

$$AB^2 = AC^2 + BC^2$$

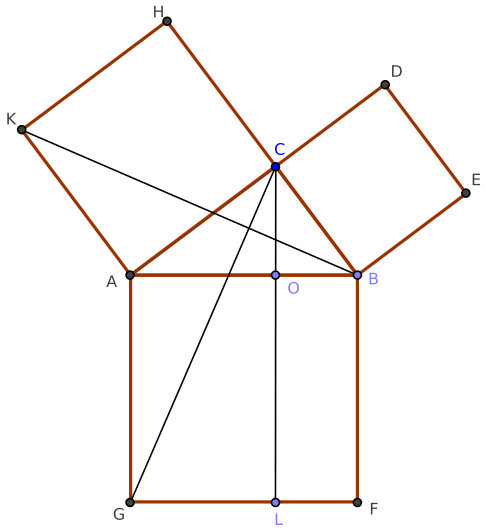

FIGURE 1.5.27. [1.47]

By [1.46], we may construct squares on sides AB, BC, and CA of $\triangle ABC$ similarly to Fig. 1.5.26. Construct segment $\overline{CL} \parallel \overline{AG}$ where L is a point on $\overline{GF}$. Also construct $\overline{CG}$ and $\overline{BK}$. Because both $\angle ACB$ and $\angle ACH$ are right angles by construction, the sum $\angle ACB + \angle ACH$ equals two right angles. Therefore $\overline{BC} \oplus \overline{CH} = \overline{BH}$ [1.14]. Similarly, $\overline{AC} \oplus \overline{CD} = \overline{AD}$.

Because $\angle BAG$ and $\angle CAK$ are each angles within a square, they are right angles. Hence,

$$\begin{aligned} \angle BAG &= \angle CAK \\ \angle BAG + \angle BAC &= \angle CAK + \angle BAC \\ \angle CAG &= \angle KAB \end{aligned}$$

Since $\boxdot BAGF$ and $\boxdot CHKA$ are squares, $BA = AG$ and $CA = AK$. Consider $\triangle CAG$ and $\triangle KAB$: since $CA = AK$, $BA = AG$, and $\angle CAG = \angle KAB$, by [1.4] $\triangle CAG \cong \triangle KAB$.

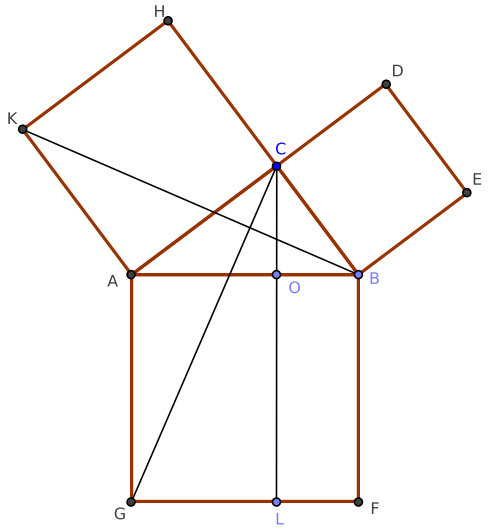

FIGURE 1.5.28. [1.47]

By [1.41], $\boxdot AGLO = 2 \cdot \triangle CAG$ because they both stand on $\overline{AG}$ and stand between the parallels $\overline{AG}$ and $\overline{CL}$. Similarly, $\boxdot CHKA = 2 \cdot \triangle KAB$ because they stand on $\overline{AK}$ and between $\overline{AK}$ and $\overline{BH}$. Since $\triangle CAG \cong \triangle KAB$, $\boxdot AGLO = \boxdot KACH$.

Similarly, it can be shown that $\boxdot OLFB = \boxdot DCBE$. Hence,

$$\begin{aligned} AB^2 &= \boxdot AGFB \\ &= \boxdot AGLO \oplus \boxdot OLFB \\ &= \boxdot KACH + \boxdot DCBE \\ &= AC^2 + BC^2 \end{aligned}$$

which proves our claim. □

REMARK. [1.47] is a special case of [6.31].

REMARK. [Cor. 10.28.1] describes a ratio which provide Pythagorean Triples[27] (three positive integers a, b, and c such that $a^2+b^2=c^2$): if a and b are positive integers and $b<a$, then Pythagorean Triples follow the ratio

$$ab : \frac{a^2-b^2}{2} : \frac{a^2+b^2}{2}$$

Alternatively:

PROOF. Construct the squares as in Fig. 1.5.28 such that $\triangle ACB$ is a right triangle where $\angle ACB$ is the right angle.

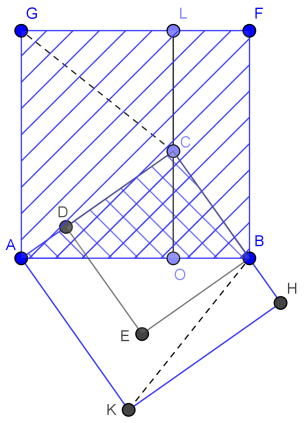

FIGURE 1.5.29. [1.47], alternate proof

Construct $\overline{CG}$ and $\overline{BK}$; through C construct $\overline{OL} \parallel \overline{AG}$. Notice that $\angle GAK = \angle GAC + \angle BAC + \angle BAK$ where $\angle BAG$ and $\angle CAK$ are right angles. It follows that:

$$\begin{aligned} \angle BAG &= \angle CAK \\ \angle BAG - \angle BAC &= \angle CAK - \angle BAC \\ \angle CAG &= \angle BAK \end{aligned}$$

Consider $\triangle CAG$ and $\triangle BAK$: $CA = AK$, $AG = AB$, and $\angle CAG = \angle BAK$; by [1.4], $\triangle CAG \cong \triangle BAK$.

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REMARK. See also https://en.wikipedia.org/wiki/Pythagorean_triple

By [1.41], we have that $\boxdot GAOL = \boxdot AKHC$. Similarly, $\boxdot LOBF = \boxdot DEBC$. $\square$

The alternative proof is shorter since it's not necessary to prove that $\overline{AC}$ and $\overline{CD}$ form one segment. Similarly, the proposition may be proven by taking any of the eight figures formed by turning the squares in all possible directions. Another simplification of the proof can be obtained by considering that the point A is such that one of the triangles $\triangle CAG$ or $\triangle BAK$ can be turned round it in its own plane until it coincides with the other; hence, they are congruent.

Exercises.

1. Prove that the square on $\overline{AC}$ is equal in area to the rectangle $\boxdot \overline{AB} \cdot \overline{AO}$, and the square on $\boxdot \overline{BC} = \boxdot \overline{AB} \cdot \overline{BO}$. (Note: $\boxdot \overline{AB} \cdot \overline{AO}$ denotes the rectangle formed by the segments $\overline{AB}$ and $\overline{AO}$.)

2. Prove that the square on $\boxdot CO = \boxdot AO \cdot OB$.

3. Prove that $AC^2 - BC^2 = AO^2 - BO^2$

4. Find a segment whose square is equal to the sum of the areas of two given squares. [See the final chapter for a solution.]

5. Given the base of a triangle and the difference of the squares of its sides, the locus of its vertex is a segment perpendicular to the base.

6. Prove that $\overline{BK} \perp \overline{CG}$. $\overline{BK}$ and $\overline{CG}$ are transverse segments.

7. If $\overline{EG}$ is constructed, then $\overline{EG}^2 = \overline{AC}^2 + 4 \cdot \overline{BC}^2$.

8. The square constructed on the sum of the sides of a right triangle exceeds the square on the hypotenuse by four times the area of the triangle (see [1.46, #3]). More generally, if the vertical angle of a triangle is equal to the angle of a regular polygon of n sides, then the regular polygon of n sides, constructed on a segment equal to the sum of its sides exceeds the area of the regular polygon of n sides constructed on the base by n times the area of the triangle.

9. If $\overline{AC}$ and $\overline{BK}$ intersect at P and a segment is constructed through P which is parallel to $\overline{BC}$, meeting $\overline{AB}$ at Q, then $\overline{CP} = \overline{PQ}$.

10. Each of the triangles $\triangle AGK$ and $\triangle BEF$ formed by joining adjacent corners of the squares is equal in area to $\triangle ABC$. [See the final chapter for a solution.]

11. Find a segment whose square is equal to the difference of the squares on two segments.

12. The square on the difference of the sides $\overline{AC}$, $\overline{CB}$ is less than the square on the hypotenuse by four times the area of the triangle.

13. If $\overline{AE}$ is connected, then the segments $\overline{AE}$, $\overline{BK}$, $\overline{CL}$ are concurrent.

14. In an equilateral triangle, three times the square on any side is equal to four times the square on the perpendicular to it from the opposite vertex.

15. We construct the square $\boxdot BEFG$ on $\overline{BE}$, a part of the side $\overline{BC}$ of a square $\boxdot ABCD$, having its side $\overline{BG}$ in the continuation of $\overline{AB}$. Divide the figure $AGFECD$ into three parts which will form a square.

16. Four times the sum of the squares on the medians which bisect the sides of a right triangle is equal to five times the square on the hypotenuse.

17. If perpendiculars fall on the sides of a polygon from any point and if we divide each side into two segments, then the sum of the squares on one set of alternate segments is equal to the sum of the squares on the remaining set.

18. The sum of the squares on segments constructed from any point to one pair of opposite angles of a rectangle is equal to the sum of the squares on the segments from the same point to the remaining pair.

19. Divide the hypotenuse of a right triangle into two parts such that the difference between their squares equals the square on one of the sides.

20. From the endpoints of the base of a triangle, let altitudes fall on the opposite sides. Prove that the sum of the rectangles contained by the sides and their lower segments is equal to the square on the base.

PROPOSITION 1.48. *THE CONVERSE OF THE PYTHAGOREAN/GOUGU THEOREM.*

Construct squares on all sides of a triangle. If the square on the longest side is equal in area to the sum of the areas of the squares on the remaining sides, then the angle opposite to the longest side is a right angle.

PROOF. Construct $\triangle ABC$ such that AB is the longest side and where

$$AB^2 = AC^2 + BC^2$$

We claim that $\angle ACB$ is a right angle.

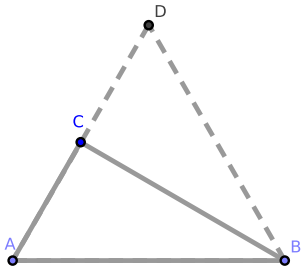


FIGURE 1.5.30. [1.48]

Construct $\overline{CD}$ such that $\overline{CD} = \overline{CA}$ [1.3] and $\overline{CD} \perp \overline{CB}$ [1.11]. Construct $\overline{BD}$. Consider $\triangle BCD$: $\angle BCD$ is a right angle by construction.

$AC = CD$ implies that $AC^2 = CD^2$, and so

$$AC^2 + CB^2 = CD^2 + CB^2$$

By [1.47] $CD^2 + CB^2 = BD^2$; by hypothesis, $AC^2 + CB^2 = AB^2$. Hence $AB^2 = BD^2$; it follows that $AB = BD$ [1.46, #1].

Consider $\triangle ACB$ and $\triangle DCB$: $AB = DB$, $AC = CD$ by construction, and each shares the side CD. By [1.8], $\triangle ACB \cong \triangle DCB$, and so $\angle ACB = \angle DCB$. But $\angle DCB$ is a right angle by construction, and so $\angle ACB$ is a right angle. $\square$

An alternate proof:

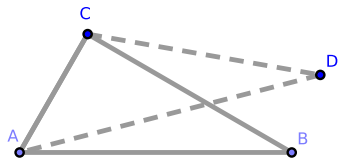


FIGURE 1.5.31. [1.48], alternative proof

PROOF. Construct $\triangle ABC$ such that $AC^2 + BC^2 = AB^2$. If $CB \not\perp CA$, construct $\overline{CD} \perp \overline{CA}$ such that $\overline{CD} = \overline{CB}$. Construct $\overline{AD}$.

Consider $\triangle ABC$ and $\triangle ADC$: as in the above proof, it can be shown that $AD = AB$. This contradicts [1.7]; it follows that, $\angle ACB$ is a right angle. □

Exam questions on chapter 1.

1. What is geometry?

2. What is geometric object?

3. Name the primary concepts of geometry. (Ans. Points, lines, surfaces, and solids.)

4. What kinds of lines exist in geometry? (Ans. Straight and curved.)

5. How is a straight line constructed? (Ans. By connecting any two collinear points.)

6. How is a curved line constructed? (Ans. By connecting any three non-collinear points.)

7. How may surfaces be divided? (Ans. Into planes and curved surfaces.)

8. How may a plane surface be constructed?

9. Why does a point have no dimensions?

10. Does a line have either width nor thickness?

11. How many dimensions does a surface possess?

12. What is plane geometry?

13. What portion of plane geometry forms the subject of this chapter?

14. What is the subject-matter of the remaining chapters?

15. How is a proposition proved indirectly?

16. What is meant by the inverse of a proposition?

17. What proposition is an instance of the Rule of Symmetry?

18. What are congruent figures?

19. What is another way to describe congruent figures? (Ans. They are identically equal.)

20. Mention all the instances of equality which are not congruence that occur in chapter 1.

21. What is the difference between the symbols denoting congruence and equality?

22. Define adjacent, exterior, interior, and alternate angles.

23. What is meant by the projection of one line on another?

24. What are meant by the medians of a triangle?

25. What is meant by the third diagonal of a quadrilateral?

26. State some propositions in chapter 1 which are particular cases of more general ones that follow.

27. What is the sum of all the exterior angles of any polygon equal to?

28. How many conditions must be given in order to construct a triangle? (Ans. Three; such as the three sides, or two sides and an angle, etc.)

Chapter 1 exercises.

1. Suppose $\triangle_1$ and $\triangle_2$ are triangles such that:

(a)$\triangle_1$ is constructed within $\triangle_2$

(b) each side of $\triangle_2$ passes through one vertex of $\triangle_1$

(c) each side of $\triangle_2$ is parallel to its opposite side in $\triangle_1$

We claim that $\triangle_2 = 4 \cdot \triangle_1$. [See the final chapter for a solution.]

2. The three altitudes of the first triangle in #1 are the altitudes at the midpoints of the sides of the second triangle.

3. Through a given point, construct a line so that the portion intercepted by the segments of a given angle are bisected at the point.

4. The three medians of a triangle are concurrent. (Note: we are proving the existence of the **centroid** of a triangle.) [See the final chapter for a solution.]

5. Construct a triangle given two sides and the median of the third side.

6. Let P =the perimeter of a triangle and S =the sum of the lengths of a triangle's medians. Prove that $\frac{3}{4} \cdot P < S < P$.

7. Construct a triangle given a side and the two medians of the remaining sides.

8. Construct a triangle given the three medians. [See the final chapter for a solution.]

9. The angle included between the perpendicular from the vertical angle of a triangle on the base and the bisector of the vertical angle is equal to half the difference of the base angles.

10. Find in two parallels two points which are equidistant from a given point and whose connecting line is parallel to a given line.

11. Construct a parallelogram given two diagonals and a side.

12. The shortest median of a triangle corresponds to the largest side. [See the final chapter for a solution.]

13. Find in two parallels two points standing opposite a right angle at a given point and which are equally distant from it.

14. The sum of the distances of any point in the base of an isosceles triangle from the equal sides is equal to the distance of either endpoint of the base from the opposite side.

15. The three perpendiculars at the midpoints of the sides of a triangle are concurrent. Hence, prove that perpendiculars from the vertices on the opposite sides are concurrent.

16. Inscribe a lozenge in a triangle having for an angle one angle of the triangle. [See the final chapter for a solution.]

17. Inscribe a square in a triangle having its base on a side of the triangle.

18. Find the locus of a point, the sum or the difference of whose distance from two fixed lines is equal to a given length.

19. The sum of the perpendiculars from any point in the interior of an equilateral triangle is equal to the perpendicular from any vertex on the opposite side.

20. Find a point in one of the sides of a triangle such that the sum of the intercepts made by the other sides on parallels constructed from the same point to these sides are equal to a given length.

21. If two angles exist such that their segments are respectively parallel, then their bisectors are either parallel or perpendicular.

22. Inscribe in a given triangle a parallelogram whose diagonals intersect at a given point.

23. Construct a quadrilateral where the four sides and the position of the midpoints of two opposite sides are given.

24. The bases of two or more triangles having a common vertex are given, both in magnitude and position, and the sum of the areas is given. Prove that the locus of the vertex is a straight line.

25. If the sum of the perpendiculars from a given point on the sides of a given polygon is given, then the locus of the point is a straight line.

26. If $\triangle ABC$ is an isosceles triangle whose equal sides are AB, AC and if $B'C'$ is any secant cutting the equal sides at B', C', such that $AB' + AC' = AB + AC$, prove that $B'C' > BC$.

27. If A, B are two given points and P is a point on a given line L, prove that the difference between AP and PB is a maximum when L bisects the angle $\angle APB$. Show that their sum is a minimum if it bisects the supplement.

28. Bisect a quadrilateral by a segment constructed from one of its vertices.

29. If $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are two parallel lines cut obliquely by $\overleftrightarrow{AB}$ and perpendicularly by $\overleftrightarrow{AC}$, and between these lines we construct BED, cutting $\overleftrightarrow{AC}$ at point E such that $\overline{ED} = 2 \cdot \overline{AB}$, prove that the angle $\angle DBC = \frac{1}{3} \cdot \angle ABC$.

30. If O is the point of concurrence of the bisectors of the angles of the triangle $\triangle ABC$, if $\overline{AO}$ is extended to intersect $\overline{BC}$ at D, and if $\overline{OE}$ is constructed from O such that $\overline{OE} \perp \overline{BC}$, prove that the $\angle BOD = \angle COE$.

31. The angle made by the bisectors of two consecutive angles of a convex quadrilateral is equal to half the sum of the remaining angles; the angle made by the bisectors of two opposite angles is equal to half the difference of the two other angles.

32. If in the construction of [1.47] we join EF, KG, then $EF^2 + KG^2 = 5 \cdot AB^2$.

33. Given the midpoints of the sides of a convex polygon of an odd number of sides, construct the polygon.

34. Trisect a quadrilateral by lines constructed from one of its angles.

35. Given the base of a triangle in magnitude and position and the sum of the sides, prove that the perpendicular at either endpoint of the base to the adjacent side and the external bisector of the vertical angle meet on a given line perpendicular to the base.

36. The bisectors of the angles of a convex quadrilateral form a quadrilateral whose opposite angles are supplemental. If the first quadrilateral is a parallelogram, the second is a rectangle; if the first is a rectangle, the second is a square.

37. Suppose that the midpoints of the sides AB, BC, CA of a triangle are respectively D, E, F and that $DG \parallel BF$ and intersects EF. Prove that the sides of the triangle $\triangle DCG$ are respectively equal to the three medians of the triangle $\triangle ABC$.

38. Find the path of a pool ball started from a given point which, after being reflected from the four sides of the table, will pass through another given point. (Assume that the ball does not enter a pocket.)

39. If two segments which bisect two angles of a triangle and are terminated by the opposite sides are equal in length, prove that the triangle is isosceles.

40. If a square is inscribed in a triangle, the rectangle under its side and the sum of the base and altitude is equal to twice the area of the triangle.

41. If AB, AC are equal sides of an isosceles triangle and if $BD \perp AC$, prove that $BC^2 = 2 \cdot AC \cdot CD$.

42. Given the base of a triangle, the difference of the base angles, and the sum or difference of the sides, construct it.

43. Given the base of a triangle, the median that bisects the base, and the area, construct it.

44. If the diagonals AC and BD of a quadrilateral $ABCD$ intersect at E and are bisected at the points F and G, then

$$4 \cdot \triangle EFG = (AEB + ECD) - (AED + EBC)$$

45. If squares are constructed on the sides of any triangle, the lines of connection of the adjacent corners are respectively:

(a) the doubles of the medians of the triangle;

(b) perpendicular to them.

CHAPTER 2

Rectangles

This chapter proves a number of elementary algebraic statements that are more familiar to us as algebraic equations. Algebra would not be developed for another thousand years after Euclid wrote "The Elements"; as such, it is more pedagogically direct to present Book II in truncated form.

Students should feel free to solve the exercises in this chapter algebraically.

Note that all Axioms, Definitions, and Mathematical Properties from chapter 1 will generally be used without being cited.

2.1. Definitions

1. If $\overline{AB}$ contains point C, then C is the *point of division* between $\overline{AC}$ and $\overline{CB}$. (It follows that the midpoint of $\overline{AB}$ is a special case of all such points of division.)

2. If $\overline{AB}$ is extended to point C, then point C is called a *point of external division*.

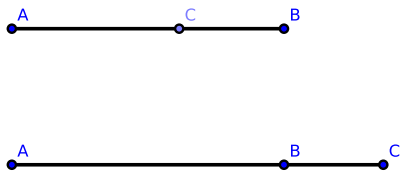


FIGURE 2.1.1. [Def. 2.1] above, [Def 2.2] below

3. A parallelogram whose angles are right angles is called a *rectangle*.

FIGURE 2.1.2. [Def. 2.3, 2.4, and 2.5]

4. A rectangle is said to be contained by any two adjacent sides: thus, $\boxdot ABCD$ is contained by sides AB and AD, or by sides AB and BC, etc.

5. The rectangle contained by two separate adjacent sides (such as AB and AD above) is the parallelogram formed by constructing a perpendicular to AB at B which is equal in length to AD and then constructing parallels.

The area of the rectangle is written $AB \cdot AD$.

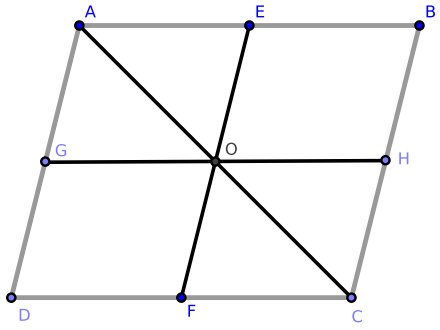


FIGURE 2.1.3. [Def. 2.6]

6. In any parallelogram, a figure which is composed of either of the parallelograms about a diagonal and the two complements is called a *gnomon* [see also 1.43]. If in Fig. 2.1.3 we remove either of the parallelograms $\boxdot AGDE$ or $\boxdot OFCH$ (but not both) from the parallelogram $\boxdot ADCB$, the remaining object would be a gnomon.

2.2. Axioms

1. A *semicircle* (half-circle) may be constructed given only its center point and a radius.

2.3. Propositions from Book II

PROPOSITION 2.1. *Suppose that two segments ($\overline{AB}$, $\overline{BD}$) which intersect at one point (B) are constructed such that one segment ($\overline{BD}$) is divided into an arbitrary but finite number of segments ($\overline{BC}$, $\overline{CE}$, $\overline{EF}$, $\overline{FD}$). Then the rectangle contained by the two segments $\overline{AB}$, $\overline{BD}$ is equal in area to the sum of the areas of the rectangles contained by $\overline{AB}$ and the subsegments of the divided segment.*

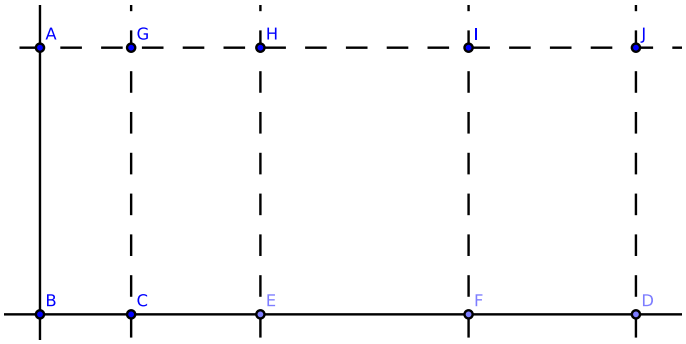


FIGURE 2.3.1. [2.1]

COROLLARY. *2.1.1. Algebraically, [2.1] states that the area*

$$AB \cdot BD = AB \cdot BC + AB \cdot CE + AB \cdot EF + AB \cdot FD$$

More generally, it states that if $y = y_1 + y_2 + ... + y_n$, then

$$\begin{aligned} xy &= x(y_1 + ... + y_n) \\ &= xy_1 + xy_2 + ... + xy_n \end{aligned}$$

[Cor. 2.1.1] is the Distributive Property from section 1.3.2 Congruence Axioms.

COROLLARY. *2.1.2. The rectangle contained by a segment and the difference of two other segments equals the difference of the rectangles contained by the segment and each of the others.*

COROLLARY. *2.1.3. The area of a triangle is equal to half the rectangle contained by its base and perpendicular.*

Exercises.

1. Prove [Cor 2.1.1].
2. Prove [Cor. 2.1.2].
3. Prove [Cor. 2.1.3].
4. An advanced project for students who studied series: prove that

$$xy = x \cdot \sum y_i$$

when $i \in \mathbb{N}$ and each $y_i > 0$.

PROPOSITION 2.2. *If a segment ($\overline{AB}$) is divided into any two subsegments at a point (C), then the square on the entire segment is equal in area to the sum of the areas of the rectangles contained by the whole and each of the subsegments ($\overline{AC}$, $\overline{CB}$).*

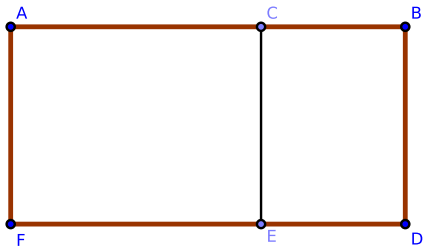


FIGURE 2.3.2. [2.2]

COROLLARY. *2.2.1. Algebraically, [2.2] is a special case of [2.1] when $n = 2$. Specifically, it states that*

$$AF \cdot FD = AF \cdot FE + AF \cdot ED$$

or if $y = y_1 + y_2$, then

$$\begin{aligned} xy &= x(y_1 + y_2) \\ &= xy_1 + xy_2 \end{aligned}$$

Exercise.

1. Prove [Cor. 1.2.1].

PROPOSITION 2.3. *If a segment ($\overline{AB}$) is divided into two subsegments (at C), the rectangle contained by the whole segment and either subsegment (CB or CF) is equal to the square on that segment together with the rectangle contained by each of the segments.*

FIGURE 2.3.3. [2.3]

COROLLARY. *2.3.1. Algebraically, [2.3] states that if $x = y + z$, then*

$$\begin{aligned} xy &= (y+z)y \\ &= y^2 + yz \end{aligned}$$

Exercise.

1. Prove [Cor. 1.3.1].

PROPOSITION 2.4. *If a segment ($\overline{AB}$) is divided into any two parts (at C), the square on the whole segment is equal in area to the sum of the areas of the squares on the subsegments ($\overline{AC}$, $\overline{CB}$) together with twice the area of their rectangle.*

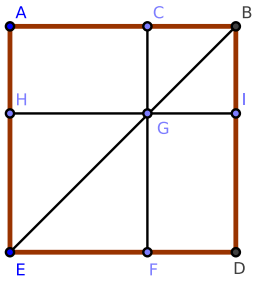


FIGURE 2.3.4. [2.4]

COROLLARY. *2.4.1. Algebraically, [2.4] states that if $x = y + z$, then*

$$\begin{aligned} x^2 &= (y+z)^2 \\ &= y^2 + 2yz + z^2 \end{aligned}$$

where $AC = y$, $CB = z$, and $AH \cdot GH = GF \cdot FI = yz$.

COROLLARY. *2.4.2. The parallelograms about the diagonal of a square are squares.*

COROLLARY. *2.4.3. If a segment is divided into any number of subsegments, the square on the whole is equal in area to the sum of the areas of the squares on all the subsegments, together with twice the sum of the areas of the rectangles contained by the several distinct pairs of subsegments.*

COROLLARY. *2.4.4. The square on a segment is equal in area to four times the square on its half.*

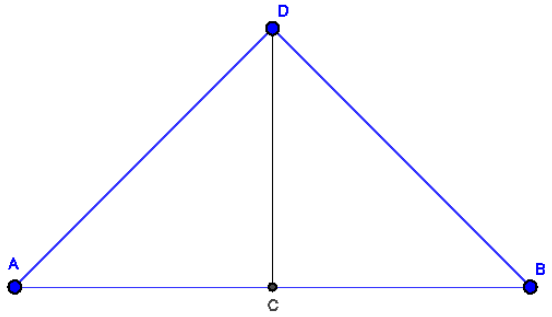


FIGURE 2.3.5. [2.4, Cor. 3]

PROOF. Suppose we have $\triangle ABD$ such that $\overline{AB} = 2 \cdot \overline{AC}$, $\overline{CD} \perp \overline{AB}$, and $\overline{CD} = \overline{AC}$. We claim that $(\overline{AB})^2 = 4 \cdot (\overline{AC})^2$.

Let $\overline{AC} = x$. Then $\overline{AB} = 2x$. It follows that

$$(\overline{AB})^2 = (2x)^2 = 4x^2 = 4 \cdot (x)^2 = 4 \cdot (\overline{AC})^2$$

This proves our claim. □

Exercises.

1. Prove [2.4] by using [2.2] and [2.3].

2. If from the right angle of a right triangle a perpendicular falls on the hypotenuse, its square equals the area of the rectangle contained by the segments of the hypotenuse. [See the final chapter for a solution.]

3. If from the hypotenuse of a right triangle subsegments are cut off equal to the adjacent sides, prove that the square on the middle segment is equal in area to twice the area of rectangle contained by the segments at either end.

4. In any right triangle, the square on the sum of the hypotenuse and perpendicular from the right angle on the hypotenuse exceeds the square on the sum of the sides by the square on the perpendicular.

5. The square on the perimeter of a right-angled triangle equals twice the rectangle contained by the sum of the hypotenuse and one side and the sum of the hypotenuse and the other side.

6. Prove [Cor. 1.4.1].

7. Prove [Cor. 1.4.2].

8. Prove [Cor. 1.4.4].

PROPOSITION 2.5. *If a segment ($\overline{AB}$) is divided into two equal parts (at C) and also into two unequal parts (at D), the rectangle contained by the unequal parts ($\overline{AD}$, $\overline{DB}$) together with the square on the part between the points of section ($\overline{CD}$) is equal in area to the square on half the line.*

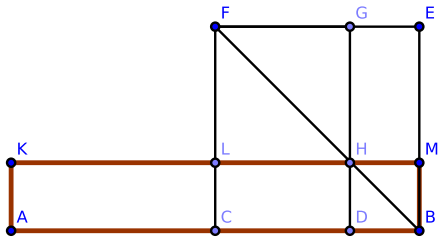


FIGURE 2.3.6. [2.5]

COROLLARY. *2.5.1. Algebraically, [2.5] states that*[1]

$$xy = \frac{(x+y)^2}{2} + \frac{(x-y)^2}{2}$$

This may also be expressed as $\overline{AD} \cdot \overline{DB} + (\overline{CD})^2 = (\overline{AC})^2 = (\overline{CB})^2$.

COROLLARY. *2.5.2. The rectangle* $\overline{AD} \cdot \overline{DB}$ *is the rectangle contained by the sum of the segments* $\overline{AC}$, $\overline{CD}$ *and their difference, and we have proved it equal to the difference between the square on* $\overline{AC}$ *and the square on* $\overline{CD}$. *Hence the difference of the squares on two segments is equal to the rectangle contained by their sum and their difference.*

COROLLARY. *2.5.3. The perimeter of the rectangle* $\overline{AH} = 2 \cdot \overline{AB}$, *and is therefore independent of the position of the point* D *on* $\overline{AB}$. *The area of the same rectangle is less than the square on half the segment by the square on the subsegment between* D *and the midpoint of the line; therefore, when* D *is the midpoint, the rectangle will have the maximum area. Hence, of all rectangles having the same perimeter, the square has the greatest area.*

Exercises.

1. Divide a given segment so that the rectangle contained by its parts has a maximum area.

[1] http://aleph0.clarku.edu/~djoyce/java/elements/bookII/propII5.html

2. Divide a given segment so that the rectangle contained by its subsegments is equal to a given square, not exceeding the square on half the given line.

3. The rectangle contained by the sum and the difference of two sides of a triangle is equal to the rectangle contained by the base and the difference of the segments of the base made by the perpendicular from the vertex.

4. The difference of the sides of a triangle is less than the difference of the segments of the base made by the perpendicular from the vertex.

5. The difference between the square on one of the equal sides of an isosceles triangle and the square on any segment constructed from the vertex to a point in the base is equal to the rectangle contained by the segments of the base.

6. The square on either side of a right triangle is equal to the rectangle contained by the sum and the difference of the hypotenuse and the other side.

7. Prove [Cor. 2.5.1].

8. Prove [Cor. 2.5.2].

9. Prove [Cor. 2.5.3].

PROPOSITION 2.6. *If a segment ($\overline{AB}$) is bisected (at C) and extended to a segment ($\overline{BD}$), the rectangle contained by the segments ($\overline{AD}$, $\overline{BD}$) made by the endpoint of the second segment (D) together with the square on half of the segment ($\overline{CB}$) equals the square on the segment between the midpoint and the endpoint of the second segment.*

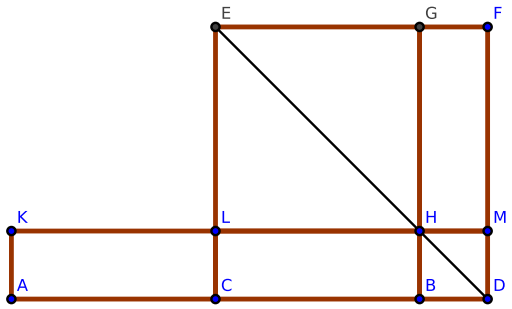


FIGURE 2.3.7. [2.6]

COROLLARY. *2.6.1. Algebraically, [2.6] states that*[2]

$$x(x-b) = (x - \frac{b}{2})^2 - (\frac{b}{2})^2$$

This may also be expressed as $\overline{AD} \cdot \overline{BD} + (\overline{CB})^2 = (\overline{CD})^2$.

Exercises.

1. Show that [2.6] is reduced to [2.5] by extending the line in the opposite direction.

2. Divide a given segment externally so that the rectangle contained by its subsegments is equal to the square on a given line.

3. Given the difference of two segments and the rectangle contained by them, find the subsegments.

4. The rectangle contained by any two segments equals the square on half the sum minus the square on half the difference.

5. Given the sum or the difference of two lines and the difference of their squares, find the lines.

[2] http://aleph0.clarku.edu/~djoyce/java/elements/bookII/propII6.html

6. If from the vertex C of an isosceles triangle a segment $\overline{CD}$ is constructed to any point in the extended base, prove that $(\overline{CD})^2 - (\overline{CB})^2 = \overline{AD} \cdot \overline{DB}$.

7. Give a common statement which will include [2.5] and [2.6]. [See the final chapter for a solution.]

8. Prove [Cor. 2.6.1].

PROPOSITION 2.7. *If a segment ($\overline{AB}$) is divided into any two parts (at C), the sum of the areas of the squares on the whole segment (AB) and either subsegment ($\overline{CB}$) equals twice the rectangle ($\overline{AB}$, $\overline{CB}$) contained by the whole segment and that subsegment, together with the square on the remaining segment.*

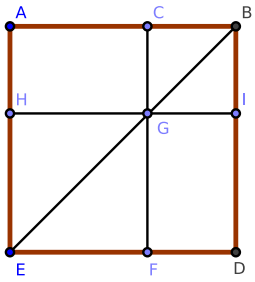


FIGURE 2.3.8. [2.7]

COROLLARY. *2.7.1. Algebraically, [2.7] states that if $x = y + z$, then*[3]

$$\begin{aligned} x^2 + z^2 &= (y+z)^2 + z^2 \\ &= y^2 + 2yz + 2z^2 \\ &= y^2 + 2z(y+z) \\ &= y^2 + 2xz \end{aligned}$$

Or,

$$(\overline{AB})^2 + (\overline{BC})^2 = 2 \cdot \overline{AB} \cdot \overline{BC} + (\overline{AC})^2$$

Equivalently, this result can be stated as $x^2 + z^2 = 2xz + (x–z)^2$.

COROLLARY. *2.7.2. Comparison of [2.4] and [2.7]:*

[2.4]: square on sum = sum of the areas of squares + twice rectangle

[2.7]: square on difference = sum of the areas of squares−twice rectangle

[3] http://aleph0.clarku.edu/~djoyce/java/elements/bookII/propII7.html

PROPOSITION 2.8. *If a segment ($\overline{AB}$) is cut arbitrarily (at C), then four times the area of the rectangle contained by the whole and one of the segments ($\overline{AB}$, $\overline{BC}$) plus the area of the square on the remaining segment ($\overline{AC}$) equals the area of the square constructed on the whole and the aforesaid segment constructed as on one segment ($\overline{AB} \oplus \overline{BC}$).*[4]

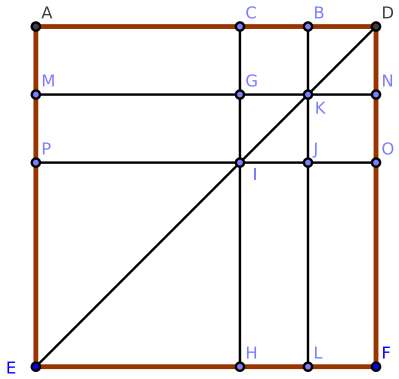


FIGURE 2.3.9. [2.8]

COROLLARY. *2.8.1. Algebraically, [2.8] states that if $x = y + z$, then*

$$\begin{aligned}
(x+y)^2 &= x^2 + 2xy + y^2 \\
&= (y+z)^2 + 2(y+z)y + y^2 \\
&= y^2 + 2yz + z^2 + 2y^2 + 2yz + y^2 \\
&= 4y^2 + 4yz + z^2 \\
&= 4y(y+z) + z^2 \\
&= 4xy + z^2 \\
&= 4xy + (x-y)^2
\end{aligned}$$

Exercises.

1. In [1.47], if EF, GK are joined, prove that $(\overline{EF})^2 - (\overline{CO})^2 = (\overline{AB} + \overline{BO})^2$.

2. In [1.47], prove that $(\overline{GK})^2 - (\overline{EF})^2 = 3 \cdot \overline{AB} \cdot (\overline{AO} - \overline{BO})$.

3. Given that the difference of two segments equals R and the area of their rectangle equals $4R^2$, find the segments.

[4] http://aleph0.clarku.edu/~djoyce/java/elements/bookII/propII8.html

PROPOSITION 2.9. *If a segment ($\overline{AB}$) is bisected (at C) and divided into two unequal segments (at D), the area of the squares on the unequal subsegments ($\overline{AD}$, $\overline{DB}$) is double the area of the squares on half the line ($\overline{AC}$) and on the segment ($\overline{CD}$) between the points of section.*

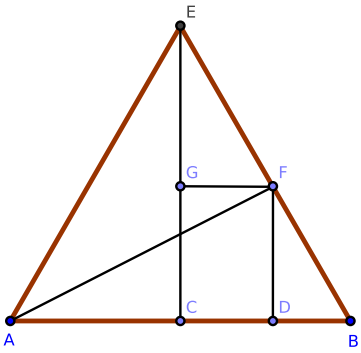


FIGURE 2.3.10. [2.9]

COROLLARY. *Algebraically, [2.9] states that*

$$(y+z)^2 + (y-z)^2 = 2(y+z)^2$$

Exercises.

1. The sum of the squares on the subsegments of a larger segment of fixed length is a minimum when it is bisected.

2. Divide a given segment internally so that the sum of the areas of the squares on the subsegments equals the area of a given square and state the limitation to its possibility.

3. If a segment $\overline{AB}$ is bisected at C and divided unequally in D, then $(\overline{AD})^2 + (\overline{DB})^2 = 2 \cdot \overline{AD} \cdot \overline{DB} + 4 \cdot (\overline{CD})^2$.

4. Twice the area of a square on the segment joining any point in the hypotenuse of a right isosceles triangle to the vertex is equal to the sum of the areas of the squares on the segments of the hypotenuse.

5. If a segment is divided into any number of subsegments, the continued product of all the parts is a maximum and the sum of the areas of their squares is a minimum when all the parts are equal.

PROPOSITION 2.10. *If a segment ($\overline{AB}$) is bisected (at C) and is extended to a segment ($\overline{AD}$), the sum of the areas of the squares on the segments ($\overline{AD}$, $\overline{DB}$) made by the endpoint (D) is equal to twice the area of the square on half the segment and twice the square on the segment between the points of that section.*

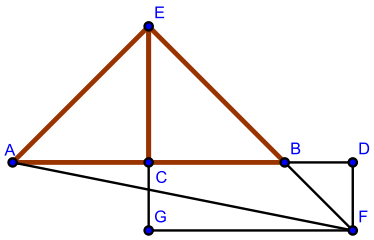


FIGURE 2.3.11. [2.10]

COROLLARY. *2.10.1. Algebraically, [2.10] states the same result as Proposition 2.9:*

$$(y+z)^2 + (y-z)^2 = 2(y+z)^2$$

COROLLARY. *2.10.2. The square on the sum of any two segments plus the square on their difference equals twice the area of the sum of their squares.*

COROLLARY. *2.10.3. The sum of the area of the squares on any two segments is equal to twice the area of the square on half the sum plus twice the square on half the difference of the lines.*

COROLLARY. *2.10.4. If a segment is cut into two unequal subsegments and also into two equal subsegments, the sum of the area of the squares on the two unequal subsegments exceeds the sum of the areas of the squares on the two equal subsegments by the sum of the areas of the squares of the two differences between the equal and unequal subsegments.*

Exercises.

1. Given the sum or the difference of any two segments and the sum of the areas of their squares, find the segments.

2. Consider $\triangle ABC$: the sum of the areas of the squares on two sides AC, CB is equal to twice the area of the square on half the base AB and twice the square on the median which bisects AB.

3. If the base of a triangle is given both in magnitude and position and the sum of the areas of the squares on the sides in magnitude, the locus of the vertex is a circle.

4. Consider $\triangle ABC$: if a point D on the base BC exists such that $(\overline{BA})^2 + (\overline{BD})^2 = (\overline{CA})^2 + (\overline{CD})^2$, prove that the midpoint of AD is equally distant from both B and C.

5. Prove [Cor. 2.10.1].

6. Prove [Cor. 2.10.2].

7. Prove [Cor. 2.10.3].

PROPOSITION 2.11. *It is possible to divide a given segment ($\overline{AB}$) into two segments (at H) such that the rectangle ($\overline{AB}$, $\overline{BH}$) contained by the segment and its subsegment is equal in area to the square on the remaining segment ($\overline{AH}$).*

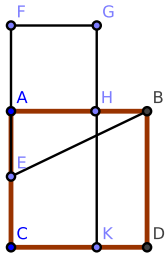


FIGURE 2.3.12. [2.11]

Definition:

1. A segment divided as in this proposition is said to be divided in "extreme and mean ratio."

COROLLARY. *2.11.1. Algebraically, [2.11] solves the equation $\overline{AB}\cdot\overline{BH} = (\overline{AH})^2$, or $a(a-x) = x^2$. Specifically,*

$$\begin{aligned} a(a-x) &= x^2 \\ a^2 - ax &= x^2 \\ x^2 + ax - a^2 &= 0 \\ &\Longrightarrow \\ x &= -\tfrac{a}{2}(1 \pm \sqrt{5}) \end{aligned}$$

Note that $\phi = \frac{1+\sqrt{5}}{2}$ is called the Golden Ratio[5].

COROLLARY. *2.11.2. The segment $\overline{CF}$ is divided in "extreme and mean ratio" at A.*

COROLLARY. *2.11.3. If from the greater segment $\overline{CA}$ of $\overline{CF}$ we take a segment equal to $\overline{AF}$, it is evident that $\overline{CA}$ will be divided into parts respectively equal to $\overline{AH}$, $\overline{HB}$. Hence, if a segment is divided in extreme and mean ratio, the greater segment will be cut in the same manner by taking on it a part equal to the less, and the less will be similarly divided by taking on it a part equal to the difference, and so on.*

[5] https://en.wikipedia.org/wiki/Golden_ratio

COROLLARY. *2.11.4. Let $\overline{AB}$ be divided in "extreme and mean ratio" at C. It is evident ([2.11], Cor. 2) that $\overline{AC} > \overline{CB}$. Cut off $\overline{CD} = \overline{CB}$. Then by ([2.11], Cor. 2), $\overline{AC}$ is cut in "extreme and mean ratio" at D, and $\overline{CD} > \overline{AD}$. Next, cut off $\overline{DE} = \overline{AD}$, and in the same manner we have $\overline{DE} > \overline{EC}$, and so on. Since $\overline{CD} > \overline{AD}$, it is evident that $\overline{CD}$ is not a common measure of $\overline{AC}$ and $\overline{CB}$, and therefore not a common measure of $\overline{AB}$ and $\overline{AC}$. Similarly, $\overline{AD}$ is not a common measure of $\overline{AC}$ and $\overline{CD}$ and so is therefore not a common measure of $\overline{AB}$ and $\overline{AC}$. Hence, no matter how far we proceed, we cannot arrive at any remainder which will be a common measure of $\overline{AB}$ and $\overline{AC}$. Hence, the parts of a line divided in "extreme and mean ratio" are incommensurable (i.e., their ratio will never be a rational number).*

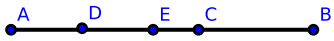


FIGURE 2.3.13. [2.11, Cor. 4]

See also [6.30] where we divide a given segment ($\overline{AB}$) into its "extreme and mean ratio"; that is, we divide $\overline{AB}$ at point C such that $\overline{AB} \cdot \overline{BC} = (\overline{AC})^2$.

Exercises.

1. The difference between the areas of the squares on the segments of a line divided in "extreme and mean ratio" is equal to the area of their rectangle.

2. In a right triangle, if the square on one side is equal in area to the rectangle contained by the hypotenuse and the other side, the hypotenuse is cut in "extreme and mean ratio" by the perpendicular on it from the right angle.

3. If $\overline{AB}$ is cut in "extreme and mean ratio" at C, prove that

   (a) $(\overline{AB})^2 + (\overline{BC})^2 = 3 \cdot (\overline{AC})^2$

   (b) $(\overline{AB} + \overline{BC})^2 = 5 \cdot (\overline{AC})^2$

   [See the final chapter for a solution.]

4. The three lines joining the pairs of points G, B; F, D; A, K, in the construction of [2.11] are parallel.

5. If $\overline{CH}$ intersects $\overline{BE}$ at O, then $\overline{AO} \perp \overline{CH}$.

6. If $\overline{CH}$ is extended, then $\overline{CH} \perp \overline{BF}$.

7. Suppose that $\triangle ABC$ is a right-angled triangle having $\overline{AB} = 2 \cdot \overline{AC}$. If $\overline{AH}$ is equal to the difference between $\overline{BC}$ and $\overline{AC}$, then $\overline{AB}$ is divided in "extreme and mean ratio" at H.

8. Prove [Cor. 2.11.1].

9. Prove [Cor. 2.11.2].

10. Prove [Cor. 2.11.3].

11. Prove [Cor. 2.11.4].

PROPOSITION 2.12. *On an obtuse triangle ($\triangle ABC$), the square on the side opposite the obtuse angle (AB) exceeds the sum of the areas of the squares on the sides containing the obtuse angle (BC, CA) by twice the area of the rectangle contained by either of them (BC) and its extension ($\overline{CD}$) to meet a perpendicular ($\overline{AD}$) on it from the opposite angle.*

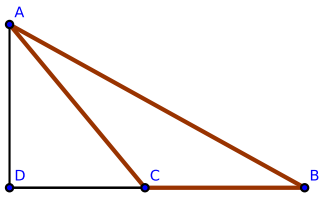


FIGURE 2.3.14. [2.12]

COROLLARY. *2.12.1. Algebraically, [2.12] states that in an obtuse triangle*

$$(\overline{AB})^2 = (\overline{AC})^2 + (\overline{BC})^2 + 2 \cdot \overline{BC} \cdot \overline{CD}$$

This is extremely close to stating the law of cosines[6]*:*

$$c^2 = a^2 + b^2 – 2ab \cdot cos(\alpha)$$

COROLLARY. *2.12.2. If perpendiculars from A and B to the opposite sides meet them in H and D, the rectangle $AC \cdot CH$ is equal in area to the rectangle $BC \cdot CD$ (or $\boxdot AC \cdot CH = \boxdot BC \cdot CD$).*

Exercises.

1. If the angle $\angle ACB$ of a triangle is equal to twice the angle of an equilateral triangle, then $AB^2 = BC^2 + CA^2 + BC \cdot CA$.

2. Suppose that $ABCD$ is a quadrilateral whose opposite angles at points B and D are right, and when $\overline{AD}$, $\overline{BC}$ are extended meet at E, prove that $\overline{AE} \cdot \overline{DE} = \overline{BE} \cdot \overline{CE}$.

3. If $\triangle ABC$ is a right triangle and $\overline{BD}$ is a perpendicular on the hypotenuse AC, prove that $\overline{AB} \cdot \overline{DC} = \overline{BD} \cdot \overline{BC}$.

4. If a segment $\overline{AB}$ is divided at C so that $(\overline{AC})^2 = 2 \cdot (\overline{BC})^2$, prove that $(\overline{AB})^2 + (\overline{BC})^2 = 2 \cdot \overline{AB} \cdot \overline{AC}$.

[6] https://en.wikipedia.org/wiki/Trigonometric_functions

5. If $\overline{AB}$ is the diameter of a semicircle, find a point C in $\overline{AB}$ such that, joining C to a fixed point D in the circumference and constructing a perpendicular $\overline{CE}$ intersecting the circumference at E, then $(\overline{CE})^2 - (\overline{CD})^2$ is equal to a given square.

6. If the square of a segment $\overline{CD}$, constructed from the angle C of an equilateral triangle $\triangle ABC$ to a point D on the extended side $\overline{AB}$ is equal in area to $2 \cdot (\overline{AB})^2$, prove that $\overline{AD}$ is cut in "extreme and mean ratio" at B.

7. Prove [Cor. 2.12.1].

8. Prove [Cor. 2.12.2].

PROPOSITION 2.13. *In any triangle ($\triangle ABC$), the square on any side opposite an acute angle (at C) is less than the sum of the squares on the sides containing that angle by twice the area of the rectangle ($\overline{BC}$, $\overline{CD}$) contained by either of them (BC) and the intercept ($\overline{CD}$) between the acute angle and the foot of the perpendicular on it from the opposite angle.*

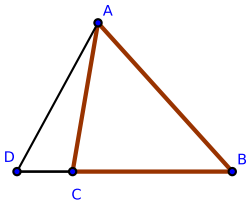


FIGURE 2.3.15. [2.13]

COROLLARY. *2.13.1. Algebraically, [2.13] states the same result as [2.12].*

Exercises.

1. If the angle at point C of the $\triangle ACB$ is equal to an angle of an equilateral triangle, then $AB^2 = AC^2 + BC^2 - AC.BC$.

2. The sum of the squares on the diagonals of a quadrilateral, together with four times the square on the line joining their midpoints, is equal to the sum of the squares on its sides.

3. Find a point C in a given extended segment AB such that $AC^2 + BC^2 = 2AC.BC$.

PROPOSITION 2.14. *CONSTRUCTION OF A SQUARE II. It is possible to construct a square equal in area to any given polygon.*

PROOF. We wish to construct a square equal in area to polygon $MNPQ$.

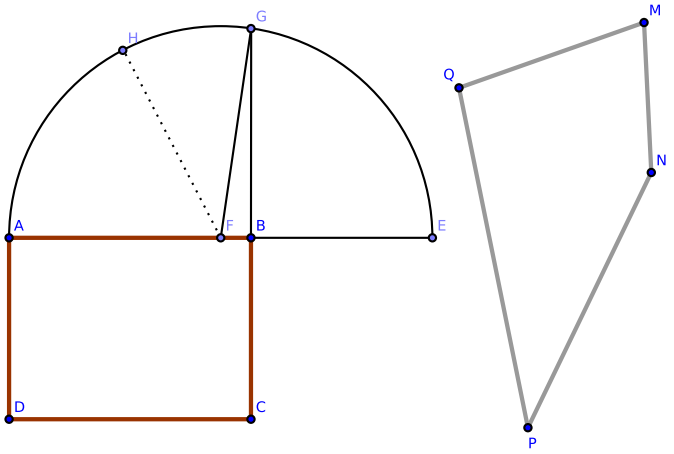


FIGURE 2.3.16. [2.14]

Construct the rectangle $\boxdot ABCD$ equal in area to $MNPQ$ [1.45]. If the adjacent sides of $\boxdot ABCD$ (AB, BC) are equal, then $\boxdot ABCD$ is a square and we have completed the construction.

Otherwise, extend AB to $\overline{AE}$ such that $\overline{BE} = \overline{BC}$. Bisect $\overline{AE}$ at F, and with F as center and $\overline{FE}$ as radius, construct the semicircle AGE. Extend $\overline{CB}$ to the semicircle at G. We claim that the square constructed on $\overline{BG}$ is equal in area to $MNPQ$.

To see this, construct $\overline{FG}$. Because $\overline{AE}$ is divided equally at F and unequally at B, by [2.5] $\overline{AB} \cdot \overline{BE} + (\overline{FB})^2 = (\overline{FE})^2$. Also, $(\overline{FE})^2 = (\overline{FG})^2$, since both are radii of semicircle AGE. And by [1.47], $(\overline{FG})^2 = (\overline{FB})^2 + (\overline{BG})^2$. Therefore,

$$\begin{aligned} \overline{AB} \cdot \overline{BE} + (\overline{FB})^2 &= (\overline{FB})^2 + (\overline{BG})^2 \\ \overline{AB} \cdot \overline{BE} &= (\overline{BG})^2 \\ \overline{AB} \cdot \overline{BC} &= (\overline{BG})^2 \\ \boxdot ABCD &= (\overline{BG})^2 \end{aligned}$$

This completes the construction. □

COROLLARY. *2.14.1. The square on the perpendicular from any point on a semicircle to the diameter is equal to the rectangle contained by the segments of the diameter.*

Exercises.

1. Given the difference of the squares on two segments and their rectangle, find the segments.

2. Prove [Cor. 2.14.1].

Chapter 2 exam questions.

1. What is the subject-matter of chapter 2? (Ans. Theory of rectangles.)

2. What is a rectangle? A gnomon?

3. What is a square inch? A square foot? A square mile? (Ans. The square constructed on a line whose length is an inch, a foot, or a mile.)

4. When is a line said to be divided internally? When externally?

5. How is the area of a rectangle determined?

6. How is a line divided so that the rectangle contained by its segments is a maximum?

7. How is the area of a parallelogram found?

8. What is the altitude of a parallelogram whose base is 65 meters and area 1430 square meters?

9. How is a segment divided when the sum of the squares on its subsegments is a minimum?

10. The area of a rectangle is $108 \cdot 60$ square meters and its perimeter is $48 \cdot 20$ linear meters. Find its dimensions.

11. What proposition in chapter 2 expresses the distributive law of multiplication?

12. On what proposition is the rule for extracting the square root founded?

13. Compare [1.47], [2.12], and [2.13].

14. If the sides of a triangle are expressed algebraically by $x^2 + 1$, $x^2 - 1$, and $2x$ units, respectively, prove that it is a right triangle.

15. How would you construct a square whose area would be exactly an acre? Give a solution using [1.47].

16. What is meant by incommensurable lines? Give an example from chapter 2.

17. Prove that a side and the diagonal of a square are incommensurable.

18. The diagonals of a lozenge are 16 and 30 meters respectively. Find the length of a side.

19. The diagonal of a rectangle is 4.25 inches, and its area is 7.50 square inches. What are its dimensions?

20. The three sides of a triangle are 8, 11, 15. Prove that it has an obtuse angle.

21. The sides of a triangle are 13, 14, 15. Find the lengths of its medians. Also find the lengths of its perpendiculars and prove that all its angles are acute.

22. If the sides of a triangle are expressed by $m^2 + n^2$, $m^2 - n^2$, and $2mn$ linear units, respectively, prove that it is right-angled.

Chapter 2 exercises.

1. The squares on the diagonals of a quadrilateral are together double the sum of the areas of the squares on the segments joining the midpoints of opposite sides.

2. If the medians of a triangle intersect at O, then $(\overline{AB})^2 + (\overline{BC})^2 + (\overline{CA})^2 = 3((\overline{OA})^2 + (\overline{OB})^2 + (\overline{OC})^2)$.

3. Through a given point O, construct three segments $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ of given lengths such that their endpoints are collinear and that $\overline{AB} = \overline{BC}$.

4. If in any quadrilateral two opposite sides are bisected, the sum of areas of the squares on the other two sides, together with the sum of areas of the squares on the diagonals, is equal to the sum of the areas of the squares on the bisected sides together with four times the area of the square on the line joining the points of bisection.

5. If squares are constructed on the sides of any triangle, the sum of the areas of the squares on the segments joining the adjacent corners is equal to three times the sum of the areas of the squares on the sides of the triangle.

6. Divide a given segment into two parts so that the rectangle contained by the whole and one segment is equal in area to any multiple of the square on the other segment.

7. If P is any point in the diameter $\overline{AB}$ of a semicircle and $\overline{CD}$ is any parallel chord, then $(\overline{CP})^2 + (\overline{PD})^2 = (\overline{AP})^2 + (\overline{PB})^2$.

8. If A, B, C, D are four collinear points taken in order, then $AB \cdot CD + BC \cdot AD = AC \cdot BD$.

9. Three times the sum of the area of the squares on the sides of any pentagon exceeds the sum of the area of the squares on its diagonals by four times the sum of the area of the squares on the segments joining the midpoints of the diagonals.

10. In any triangle, three times the sum of the area of the squares on the sides is equal to four times the sum of the area of the squares on the medians.

11. If perpendiculars are constructed from the vertices of a square to any line, the sum of the squares area of the on the perpendiculars from one pair of opposite angles exceeds twice the area of the rectangle of the perpendiculars from the other pair by the area of the square.

12. If the base AB of a triangle is divided at D such that $m \cdot \overline{AD} = n \cdot \overline{BD}$, then $m \cdot (\overline{AC})^2 + n \cdot (\overline{BC})^2 = m \cdot (\overline{AD})^2 + n \cdot (\overline{DB})^2 + (m+n) \cdot (\overline{CD})^2$.

13. If the point D is taken on the extended segment $\overline{AB}$ such that $m \cdot \overline{AD} = n \cdot \overline{BD}$, then $m \cdot (\overline{AC})^2 - n \cdot (\overline{BC})^2 = m \cdot (\overline{AD})^2 - n \cdot (\overline{DB})^2 + (m-n) \cdot (\overline{CD})^2$.

14. Given the base of a triangle in magnitude and position as well as the sum or the difference of m times the square on one side and n times the square on the other side in magnitude, then the locus of the vertex is a circle.

15. Any rectangle is equal in area to half the rectangle contained by the diagonals of squares constructed on its adjacent sides. [See the final chapter for a solution.]

16. If A, B, C, ... are any finite number of fixed points and P a movable point, find the locus of P if $(\overline{AP})^2 + (\overline{BP})^2 + (\overline{CP})^2 + ...$ is given.

17. If the area of a rectangle is given, its perimeter is a minimum when it is a square.

18. Construct equilateral triangles on subsegments $\overline{AC}$, $\overline{CB}$ of segment $\overline{AB}$. Prove that if D, D_1 are the centers of circles constructed about these triangles, then $6 \cdot D \cdot D_1^2 = (\overline{AB})^2 + (\overline{AC})^2 + (\overline{CB})^2$.

19. If a, b denote the sides of a right triangle about the right angle and p denotes the perpendicular from the right angle on the hypotenuse, then $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

20. If upon the greater subsegment $\overline{AB}$ of a segment $\overline{AC}$ which is divided in extreme and mean ratio, an equilateral triangle $\triangle ABD$ is constructed and $\overline{CD}$ is joined, then $(\overline{CD})^2 = 2 \cdot (\overline{AB})^2$.

21. If a variable line, whose endpoints rest on the circumferences of two given concentric circles, stands opposite a right angle at any fixed point, then the locus of its midpoint is a circle.

CHAPTER 3

Circles

Axioms and Mathematical Properties from chapters 1 and 2 will be assumed and not usually cited. This will be a rule that we will apply to subsequent chapters, *mutatis mutandis*.

3.1. Definitions

1. *Equal circles* are circles with equal radii. (Note: This is a theorem, and not a definition. If two circles have equal radii, they are evidently congruent figures and therefore equal in all aspects. Using this method to prove the theorem, [3.26]-[3.29] follow immediately.)

2. A *chord* of a circle is a segment which intersects two points of the circle's circumference. If the chord is extended in both directions, then this line is called a *secant*, and each of the parts into which a secant divides the circumference is called an *arc*—the larger is called the *major conjugate arc*, and the smaller is called the *minor conjugate arc*.

3. A segment, ray, or straight line is said to *touch* a circle when it intersects the circumference of a circle at one and only one point. The segment, ray, or straight line is called a *tangent* to the circle, and the point where it touches the circumference is called the *point of intersection*.

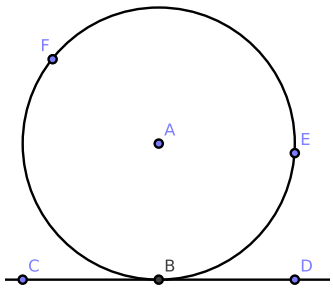


FIGURE 3.1.1. [Def. 3.3] $\overleftrightarrow{CD}$ touches $\bigcirc A$ at B; or, $\overleftrightarrow{CD}$ is tangent to $\bigcirc A$ and B is the point of intersection between $\bigcirc A$ and $\overleftrightarrow{CD}$.

REMARK. Modern geometry no longer uses Euclid's definitions for curves, tangents, etc. However, it would be far easier to write a new geometry textbook from first principles rather than attempt to update each of Euclid's definitions and then revise the propositions accordingly. All the same, Euclid's powerful presentation of complex ideas from simple axioms remains a model for how mathematics should be approached, and students who attempt to master Euclid's Elements will find 21st century mathematics more straightforward by doing so despite its not-quite-correct definitions.

4. Circles are said to touch one another when they intersect at one and only one point. There are two types of contact:

a) When one circle is external to the other.

b) When one circle is internal to the other.

When circles intersect at two points, the intersection may be referred to as a *cut*.

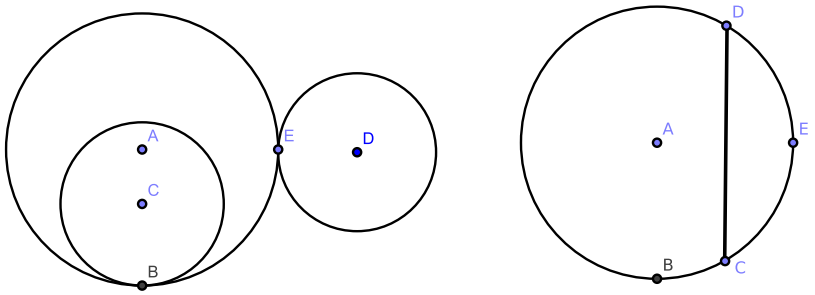


FIGURE 3.1.2. **On the left:** [Def. 3.4] The circles $\bigcirc A$ and $\bigcirc D$ touch externally at E, while the circles $\bigcirc C$ and $\bigcirc A$ touch internally at B. **On the right:** [Def. 3.5] The chord $\overline{CD}$ of the circle $\bigcirc A$ divides the circle itself into segments DEC and DBC. Segment DEC is bounded by chord $\overline{CD}$ and arc DEC, and segment DBC is bounded by chord $\overline{CD}$ and arc DBC.

5. A *segment of a circle* is a two-dimensional figure bounded by a chord and an arc whose boundary points include the endpoints of the chord.

6. Chords are said to be equally distant from the center when the perpendiculars constructed to them from the center are equal in length.

7. The *angle in the segment* is the rectilinear angle contained between two chords which intersect at the same endpoint on the circumference of a circle. In Fig. 3.1.3, $\angle DCE$ is an angle in the segment. See also [3.21].

8. The *angle of a segment* is the non-rectilinear angle contained between its chord and the tangent at either endpoint. In Fig. 3.1.3, the arc DEC is the angle of segment DEC. These angles only appeared in the original proof to [3.16].

9. An angle in a segment is said to *stand* on its conjugate arc.

10. Similar arcs are those that contain equal angles.

11. A sector of a circle is formed by two radii and the arc that is included between them. In Fig. 3.1.3, $\bigcirc A$, radius $\overline{AD}$, and radius $\overline{AC}$ form the sectors $DACE$ and $DACB$.

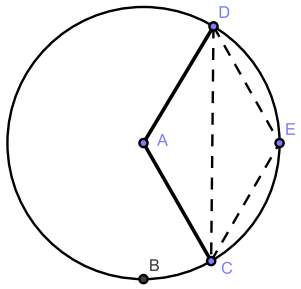


FIGURE 3.1.3. [Def. 3.8], [Def. 3.9], and [Def. 3.11]

12. *Concentric circles* are those which have the same center point.

13. Points which lie on the circumference of a circle are called *concyclic*.

14. A *cyclic quadrilateral* is a quadrilateral which is inscribed in a circle.

15. A modern definition on an angle[1]:

In geometry, an *angle* is the figure formed by two rays called the sides of the angle which share a common endpoint called the vertex of the angle. This measure is the ratio of the length of a circular arc to its radius, where the arc is centered at the vertex and delimited by the sides.

[1] `http://en.wikipedia.org/wiki/Angle`

The size of a geometric angle is usually characterized by the magnitude of the smallest rotation that maps one of the rays into the other. Angles that have the same size are called *congruent angles*.

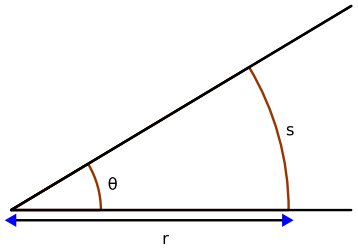


FIGURE 3.1.4. The measure of angle θ is the quotient of s and r.

In order to measure an angle θ, a circular arc centered at the vertex of the angle is constructed, e.g., with a pair of compasses. The length of the arc is then divided by the radius of the arc r, and possibly multiplied by a scaling constant k (which depends on the units of measurement that are chosen):

$$\theta = ks/r$$

The value of θ thus defined is independent of the size of the circle: if the length of the radius is changed, then the arc length changes in the same proportion, and so the ratio s/r is unaltered.

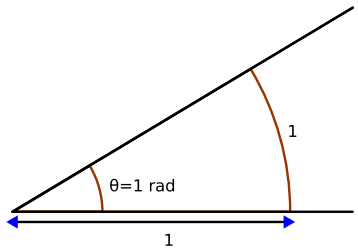


FIGURE 3.1.5. $\theta = s/r$ rad $= 1$ rad.

A number of units are used to represent angles: the radian and the degree are by far the most commonly used.

Most units of angular measurement are defined such that one turn (i.e. one full circle) is equal to n units, for some whole number n. In the case of degrees, $n = 360$. A turn of n units is obtained by setting $k = \frac{n}{2\pi}$ in the formula above.

The radian is the angle subtended by an arc of a circle (that is, the angle standing opposite the arc of a circle) that has the same length as the circle's radius. The case of radian for the formula given earlier, a radian of $n = 2\pi$ units is obtained by setting $k = \frac{2\pi}{2\pi} = 1$. One turn is 2π radians, and one radian is $\frac{180}{\pi}$ degrees, or about 57.2958 degrees. The radian is abbreviated rad, though this symbol is often omitted in mathematical texts, where radians are assumed unless specified otherwise. When radians are used angles are considered as dimensionless. The radian is used in virtually all mathematical work beyond simple practical geometry, due, for example, to the pleasing and "natural" properties that the trigonometric functions display when their arguments are in radians. The radian is the (derived) unit of angular measurement in the SI system.

The degree, written as a small superscript circle ($^\circ$), is 1/360 of a turn, so one turn is 360°. Fractions of a degree may be written in normal decimal notation (e.g. 3.5° for three and a half degrees), but the "minute" and "second" sexagesimal subunits of the "degree-minute-second" system are also in use, especially for geographical coordinates and in astronomy and ballistics.

Although the definition of the measurement of an angle does not support the concept of a negative angle, it is frequently useful to impose a convention that allows positive and negative angular values to represent orientations and/or rotations in opposite directions relative to some reference.

In a two-dimensional Cartesian coordinate system, an angle is typically defined by its two sides, with its vertex at the origin. The initial side is on the positive x-axis, while the other side or terminal side is defined by the measure from the initial side in radians, degrees, or turns. Positive angles represent rotations toward the positive y-axis, and negative angles represent rotations toward the negative y-axis. When Cartesian coordinates are represented by standard position, defined by the x-axis rightward and the y-axis upward, positive rotations are anticlockwise and negative rotations are clockwise.

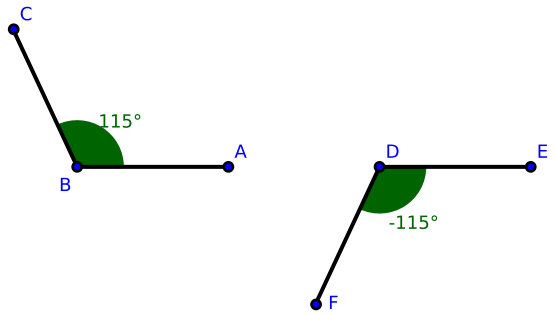


FIGURE 3.1.6. $\angle CBA$ measured as a positive angle, $\angle EDF$ measured as a negative angle

3.2. Propositions from Book III

PROPOSITION 3.1. *THE CENTER OF A CIRCLE I.*

It is possible to locate the center of a circle.

PROOF. Construct a circle and take any two points A, B on the circumference. Construct $\overline{AB}$ and bisect $\overline{AB}$ at C. Construct $\overline{CD} \perp \overline{AB}$ and extend $\overline{CD}$ to intersect the circumference at E. Bisect $\overline{DE}$ at F . We claim that F is the center of the circle.

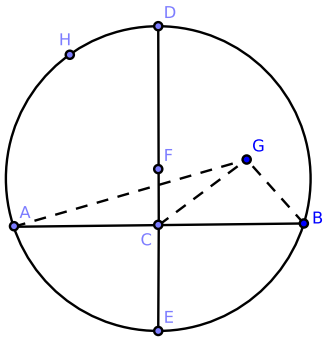


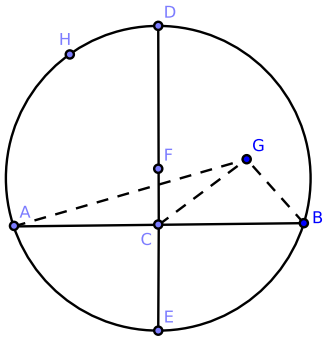
FIGURE 3.2.1. [3.1]

Suppose instead that point G, which does not lie on chord $\overline{DE}$, is the center of the circle. Construct $\overline{GA}$, $\overline{GC}$, and $\overline{GB}$. Notice that $\angle ACG = \angle ACD + \angle DCG$; clearly, $\angle DCG > 0$.

Consider $\triangle ACG$ and $\triangle BCG$: we have $AC = CB$ by construction, $GA = GB$ (since they are radii by hypothesis), and side CG is shared in common. By [1.8], we have that $\angle ACG = \angle BCG$; therefore, each angle is a right angle. But $\angle ACD$ is right by construction; therefore $\angle ACG = \angle ACD$ and $\angle DCG = 0$, a contradiction.

Hence, no point can be the center other than a point on $\overline{DE}$. Since all radii are equal in length and $\overline{FE} = \overline{FD}$, it follows that F, the midpoint of $\overline{DE}$, is the center of $\bigcirc F$. This proves our claim. □

Alternatively:



PROOF. Consider the objects constructed in the proof above: because $\overline{ED} \perp \overline{AB}$ and $\overline{ED}$ bisects $\overline{AB}$, every point equally distant from the points A and B must lie on $\overline{ED}$ [1.10, #2]. Since the center is equally distant from A and B, the center must lie on $\overline{ED}$. And since the center must also lie equally distant from E and D, the center is the midpoint of $\overline{ED}$. □

COROLLARY. *3.1.1. The line, ray, or segment which bisects any chord of a circle perpendicularly passes through the center of the circle.*

COROLLARY. *3.1.2. The locus of the centers of the circles which pass through two fixed points is the line bisecting at right angles the line that connects the two points.*

COROLLARY. *3.1.3. If A, B, C are three points on the circumference of a circle, the lines which perpendicularly bisect the chords $\overline{AB}$, $\overline{BC}$ will intersect at the center of the circle.*

Exercises.

1. Prove [Cor. 3.1.1].
2. Prove [Cor. 3.1.2].
3. Prove [Cor. 3.1.3].

PROPOSITION 3.2. *POINTS ON A LINE INSIDE AND OUTSIDE A CIRCLE.*

If any two points are chosen from the circumference of a circle and a line is constructed on those points, then:

(1) The points between the endpoints on the circumference form a chord (i.e., they lie inside the circle).

(2) The remaining points of the line lie outside the circle.

PROOF. Construct $\bigcirc C$ where A and B are arbitrary points on the circumference of $\bigcirc C$. Also construct $\overleftrightarrow{AB}$. We claim that:

(1) $\overline{AB}$ lies on the circumference of $\bigcirc C$ or within $\bigcirc C$.

(2) All points of $\overleftrightarrow{AB} - \overline{AB}$ lie outside of the circle.

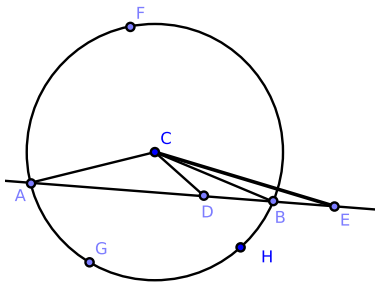


FIGURE 3.2.2. [3.2]

Take any point D on $\overline{AB}$ and construct $\overline{CA}$, $\overline{CD}$, and $\overline{CB}$. Notice that $\angle ADC > \angle ABC$ by [1.16]; however, $\angle ABC = \angle BAC$ because $\triangle CAB$ is isosceles [1.5]. Therefore, $\angle ADC > \angle BAC = \angle DAC$. By [1.29], $\overline{AC} > \overline{CD}$ [1.29], and so $\overline{CD}$ is less than the radius of $\bigcirc C$.

Consequently, D must lie within the circle [Def. 1.23]. Similarly, every other point between A and B lies within $\bigcirc C$. Finally, since A and B are points in the circumference of $\bigcirc C$, $\overline{AB}$ is a chord. This proves claim 1.

Let E be any point on $\overrightarrow{AE}$ such that $\overline{EA} > \overline{BA}$, and construct $\overline{CE}$. (Any point may also be chosen on $\overrightarrow{BA}$, *mutatis mutandis*.) By [1.16], $\angle ABC > \angle AEC$; by the above, $\angle CAE > \angle AEC$. It follows that in $\triangle ACE$, $\overline{CE} > \overline{CA}$, and so the point E lies outside $\bigcirc C$. This proves claim 2, which completes the proof. $\square$

COROLLARY. *3.2.1. Three collinear points cannot be concyclic.*

COROLLARY. *3.2.2. A straight line, ray, or segment cannot intersect a circle at more than two points.*

COROLLARY. *3.2.3. The circumference of a circle is everywhere concave towards the center.*

Exercises.

1. Prove [Cor. 3.1.1].
2. Prove [Cor. 3.1.2].
3. Prove [Cor. 3.1.3].

PROPOSITION 3.3. *CHORDS I.*

Suppose there exist two chords of a circle, one of which passes through the center of the circle. The chord which does not pass through the center is bisected by the chord that passes through the center if and only if the chords are perpendiculars.

PROOF. Construct $\bigcirc O$ with chords $\overline{AB}$ and $\overline{CD}$ where $\overline{AB}$ contains the center of $\bigcirc O$. We claim that $\overline{AB}$ bisects $\overline{CD}$ if and only if $\overline{AB} \perp \overline{CD}$.

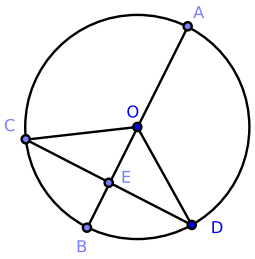


FIGURE 3.2.3. [3.3]

Suppose that $\overline{AB}$ bisects $\overline{CD}$. Construct $\overline{OC}$ and $\overline{OD}$, and consider $\triangle CEO$ and $\triangle DEO$: $\overline{CE} = \overline{ED}$ by hypothesis, $\overline{OC} = \overline{OD}$ since each are radii of $\bigcirc O$, and both triangles have $\overline{EO}$ in common. By [1.8], $\angle CEO = \angle DEO$; since they are also adjacent angles, each is a right angle, and therefore $\overline{AB} \perp \overline{CD}$.

Now suppose that $\overline{AB} \perp \overline{CD}$. Because $\overline{OC} = \overline{OD}$, $\triangle CDO$ is isosceles; by [1.5], $\angle OCD = \angle ODC$.

Consider $\triangle OEC$ and $\triangle OED$: $\angle OCD = \angle ODC$, $\angle CEO = \angle DEO$ since $\overline{AB} \perp \overline{CD}$, and they share side $\overline{EO}$. By [1.26], $\triangle OEC \cong \triangle OED$, and so $\overline{CE} = \overline{ED}$. Since $\overline{CD} = \overline{CE} \oplus \overline{ED}$, we have that $\overline{CD}$ is bisected at E by $\overline{AB}$.

This proves our claim. □

The second part of the proposition may be proved another way:

PROOF. By [1.47], we have that

$$\begin{aligned} (\overline{OC})^2 &= (\overline{OE})^2 + (\overline{EC})^2 \\ (\overline{OD})^2 &= (\overline{OE})^2 + (\overline{ED})^2 \end{aligned}$$

Since $\overline{OC} = \overline{OD}$, we also have that $(\overline{OC})^2 = (\overline{OD})^2$, and it follows that $(\overline{EC})^2 = (\overline{ED})^2$. Therefore, $\overline{EC} = \overline{ED}$. □

COROLLARY. *3.3.1. The line which bisects perpendicularly one of two parallel chords of a circle bisects the other perpendicularly.*

COROLLARY. *3.3.2. The locus of the midpoints of a system of parallel chords of a circle is the diameter of the circle perpendicular to them all.*

COROLLARY. *3.3.3. If a line intersects two concentric circles, its intercepts between the circles are equal in length.*

COROLLARY. *3.3.4. The line connecting the centers of two intersecting circles bisects their common chord perpendicularly.*

Observation: [3.1], [3.3], and [3.3, Cor. 1] are related such that if any one of them is proved directly, then the other two follow by the Rule of Symmetry.

Exercises.

1. If a chord of a circle stands opposite a right angle at a given point, the locus of its midpoint is a circle.

2. Prove [3.3, Cor. 1].

3. Prove [3.3, Cor. 2].

4. Prove [3.3, Cor. 3].

5. Prove [3.3, Cor. 4]. [See the final chapter for a solution.]

PROPOSITION 3.4. *CHORDS II.*

Suppose we have two chords of a circle, neither of which is a diameter. At most, one chord can bisect the other.

PROOF. The hypothesis and claim are stated above. We shall prove this claim using two cases.

Case 1: construct $\bigcirc O$ with chords $\overline{AB}$ and $\overline{CD}$ neither of which are diameters and which intersect at E. We claim that $\overline{AB}$ and $\overline{CD}$ do not bisect each other.

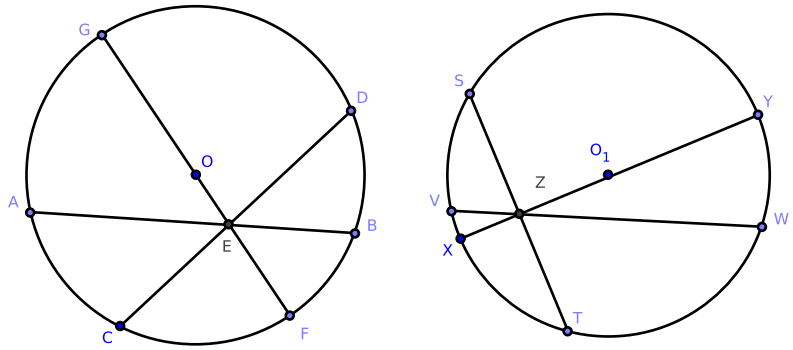


FIGURE 3.2.4. [3.4]

Since $\overline{AB}$ and $\overline{CD}$ do not contain the point O, we may construct $\overline{GOEF}$, a diameter of $\bigcirc O$. Notice that $\angle CEA > 0$.

Suppose that $\overline{AE} = \overline{EB}$ and $\overline{CE} = \overline{ED}$. Suppose that $\overline{GF}$ bisects $\overline{AB}$; by [3.3], $\overline{OE} \perp \overline{AB}$. Similarly, $\overline{OE} \perp \overline{CD}$. Hence, $\angle CEO = \angle AEO$ where $\angle CEO = \angle CEA + \angle AEO$; or, $\angle CEA = 0$, a contradiction. Therefore, $\overline{AB}$ and $\overline{CD}$ do not bisect each other.

Case 2: Construct $\bigcirc O_1$ with diameter $\overline{XY}$ and chords $\overline{VW}$ and $\overline{ST}$ where $\overline{ST} \perp \overline{XY}$ and $\overline{VW}$ is not a diameter. Notice that $\angle XZS = \angle XZV + \angle VZS$ and $\angle XZV > 0$.

Suppose that $\overline{VZ} = \overline{ZW}$ and $\overline{XZ} = \overline{ZY}$. By [3.3], $\overline{XY} \perp \overline{VW}$. It follows that $\overline{ST} \perp \overline{VW}$, and so $\angle VZS =$one right angle. But $\angle XZS =$one right angle, and so $\angle XZV = 0$, a contradiction. Therefore, $\overline{VW}$ and $\overline{XY}$ do not bisect each other. □

COROLLARY. *3.4.1. If two chords of a circle bisect each other, they are both diameters.*

PROPOSITION 3.5. *NON-CONCENTRIC CIRCLES I.*

If two circles intersect each other at two points, then they are not concentric.

PROOF. Construct $\bigcirc ABC$ and $\bigcirc ABD$ which intersect at A and B; we claim that $\bigcirc ABC$ and $\bigcirc ABD$ are not concentric.

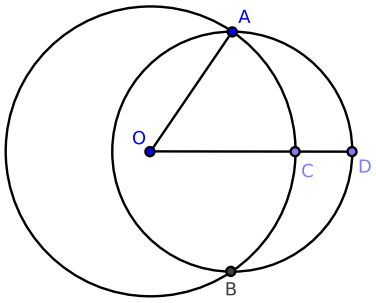


FIGURE 3.2.5. [3.5]

Suppose that $\bigcirc ABC$ and$\bigcirc ABD$ share a common center, O. Construct $\overline{OA}$ and $\overline{OCD}$ where A, B, and D are distinct. Notice that $\overline{CD} > 0$.

Because O is the center of the circle $\bigcirc ABC$, $\overline{OA} = \overline{OC}$. Because O is the center of the circle $\bigcirc ABD$, $\overline{OA} = \overline{OD}$; hence, $\overline{OD} = \overline{OC}$ where $\overline{OD} = \overline{OC} \oplus \overline{CD}$. It follows that $\overline{CD} = 0$, a contradiction. Therefore, $\bigcirc ABC$ and $\bigcirc ABD$ are not concentric. □

Exercises.

1. Two circles cannot have three points in common without coinciding. [See the final chapter for a solution.]

PROPOSITION 3.6. *NON-CONCENTRIC CIRCLES II.*

If one circle intersects another circle internally at one and only one point, then the circles are not concentric.

PROOF. Construct $\bigcirc ADE$ and $\bigcirc ABC$ such that $\bigcirc ABC$ intersects $\bigcirc ADE$ internally at A. We claim that $\bigcirc ADE$ and $\bigcirc ABC$ are not concentric.

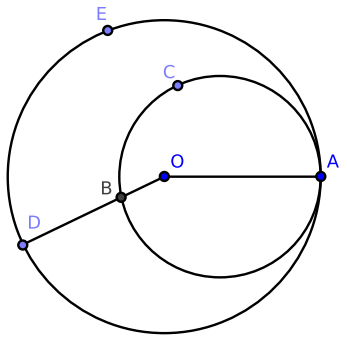


FIGURE 3.2.6. [3.6]

Suppose that $\bigcirc ADE$ and $\bigcirc ABC$ are concentric, and let O be the center of each circle. Construct $\overline{OA}$ and $\overline{OBD}$. Notice that $\overline{BD} > 0$.

Because O is the center of each circle by hypothesis, $\overline{OA} = \overline{OB}$ and $\overline{OA} = \overline{OD}$; therefore, $\overline{OB} = \overline{OD}$ and $\overline{OB} \oplus \overline{BD} = \overline{OD}$. Hence, $\overline{BD} = 0$, a contradiction. Therefore, the circles are not concentric. □

PROPOSITION 3.7. *UNIQUENESS OF SEGMENT LENGTHS FROM A POINT ON THE DIAMETER OTHER THAN THE CENTER.*

Choose any point on the diameter of a circle other than the center, and from that point construct a finite number of segments to the circumference. Then:

(1) The longest segment constructed will contain the center of the circle.

(2) The shortest segment constructed will form a diameter with the longest segment.

(3) As for the remaining segments, those with endpoints on the circumference nearer to the endpoint of the longest segment constructed will be longer than segments with endpoints farther from the endpoint on the circumference of the longest segment.

(4) Two and only two equal segments can be constructed from each point to the circumference, one on each side of the diameter.

PROOF. Construct $\bigcirc O$ with point P on diameter $\overline{AE}$ such that O and P are distinct. Construct a finite number of segments from P to the circumference ($\overline{PA}$, $\overline{PB}$, $\overline{PC}$, etc.). Notice that $\overline{PA}$ is a segment on the diameter. We will make and prove four claims.

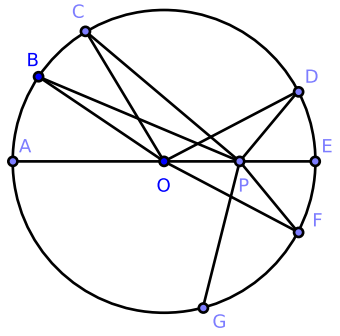


FIGURE 3.2.7. [3.7] $\bigcirc EAG$

1. The longest segment, $\overline{PA}$, is the segment which passes through O.

Construct $\overline{OB}$ where B is a point on $\bigcirc O$. Clearly, $\overline{OA} = \overline{OB}$. From this we obtain $\overline{PA} = \overline{OA} + \overline{OP} = \overline{OB} + \overline{OP}$. Consider $\triangle OPB$: since $\overline{OB} + \overline{OP} > \overline{PB}$ by [1.20], it follows that $\overline{PA} > \overline{PB}$. Hence, $\overline{PA}$ is the longest segment that can be constructed by this method; also by construction, $\overline{PA}$ contains O.

2. The extension of $\overline{PA}$ in the opposite direction, $\overline{PE}$, is the shortest segment of all such constructed segments.

Construct $\overline{OD}$ and consider $\triangle OPD$: by [1.20], $\overline{OP} + \overline{PD} > \overline{OD}$. Since, $\overline{OD} = \overline{OE} = \overline{OP} + \overline{PE}$, it follows that

$$\begin{aligned} \overline{OP} + \overline{PD} &> \overline{OP} + \overline{PE} \\ \overline{PD} &> \overline{PE} \end{aligned}$$

Since this inequality holds for any segment constructed using this method, $\overline{PE}$ is the shortest segment of all such constructed segments.

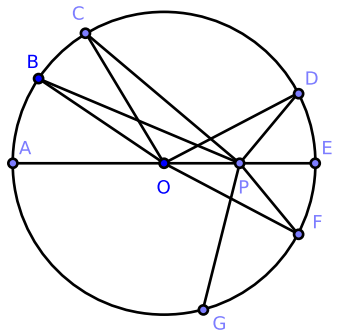


FIGURE 3.2.8. [3.7] $\bigcirc EAG$

3. As for the remaining segments, those with endpoints on the circumference nearer to the endpoint of the longest segment constructed ($\overline{PA}$) will be longer than segments with endpoints farther from the endpoint on the circumference of the longest segment (i.e., $\overline{PA} > \overline{PB} > \overline{PC} > \overline{PD}$).

Construct $\overline{OC}$, and consider $\triangle POB$ and $\triangle POC$: $\overline{OB} = \overline{OC}$ and share side $\overline{OP}$ in common. Since $\angle POB = \angle POC + \angle BOC$, we have that $\angle POB > \angle POC$. By [1.24], $\overline{PB} > \overline{PC}$. Similarly, $\overline{PC} > \overline{PD}$.

4. Two and only two segments making equal angles with the diameter and standing on opposite sides of the diameter are equal in length (i.e., $\overline{PD} = \overline{PF}$).

At O, construct $\angle POF = \angle POD$ and construct $\overline{PF}$. Consider $\triangle POD$ and $\triangle POF$: $\overline{OD} = \overline{OF}$, each shares side $\overline{OP}$, and $\angle POD = \angle POF$ by construction. By [1.4], $\triangle POD \cong \triangle POF$, and so $\angle OPF = \angle OPD$ and $\overline{PD} = \overline{PF}$.

We claim that a third segment cannot be constructed from P equal to $\overline{PD} = \overline{PF}$. Suppose this were possible and let $\overline{PG} = \overline{PD}$. Then $\overline{PG} = \overline{PF}$, which contradicts claim 3 above.

This proves our claim. □

COROLLARY. *3.7.1. If two equal segments $\overline{PD}$, $\overline{PF}$ are constructed from a point P to the circumference of a circle, the diameter through P bisects the angle $\angle DPF$ formed by these segments.*

COROLLARY. *3.7.2. If P is the common center of circles whose radii are $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, $\overline{PD}$, etc., then:*

(1) The circle whose radius is the maximum segment ($\bigcirc P$ with radius $\overline{PA}$) lies outside $\bigcirc O$ and intersects it at A [Def. 3.4].

(2) The circle whose radius is the minimum segment ($\bigcirc P$ with radius $\overline{PE}$) lies inside $\bigcirc O$ and intersects it at E.

(3) A circle having any of the remaining radii (such as $\overline{PD}$) cuts $\bigcirc O$ at two points (such as D, F).

Exercises.

1. Prove [Cor. 3.7.1].

2. Prove [Cor. 3.7.2].

REMARK. [3.7] is a good illustration of the following important definition: if a geometrical magnitude varies its position continuously according to any well-defined relationship, and if it retains the same value throughout, it is said to be a constant (such as the radius of a fixed circle).

But if a magnitude increases for some time and then begins to decrease, it is said to be a maximum when the increase stops. Therefore in the previous figure, $\overline{PA}$, which we suppose to revolve around P and meet the circle, is a maximum.

Again, if it decreases for some time, and then begins to increase, it is a minimum at the beginning of the increase. Thus $\overline{PE}$, which we suppose as before to revolve around P and meet the circle, is a minimum. [3.8] will provide other illustrations of this concept.

PROPOSITION 3.8. *SEGMENT LENGTHS FROM A POINT OUTSIDE THE CIRCLE AND THEIR UNIQUENESS.*

Suppose a point is chosen outside of a circle and from that point segments are constructed such that they intersect the circumference of the circle at two points, one on the "outer" or convex side of the circumference and one on the "inner" or concave side of the circumference. Let one segment be constructed which intersects the center of the circle and the others all within the same semicircle but not through the center of the circle. Then:

(1) The maximum segment passes through the center.

(2) The segments nearer to the segment through the center are greater in length than those which are farther away.

(3) If segments are constructed to the convex circumference, the minimum segment is that which passes through the center when extended.

(4) Of the other segments, that which is nearer to the minimum is smaller than one more farther out.

(5) From the given point outside of the circle, there can be constructed two equal segments to the concave or the convex circumference, both of which make equal angles with the line passing through the center.

(6) Three or more equal segments cannot be constructed from the given point outside the circle to either circumference.

PROOF. Construct $\bigcirc O$ and point P outside of $\bigcirc O$. We prove each claim separately.

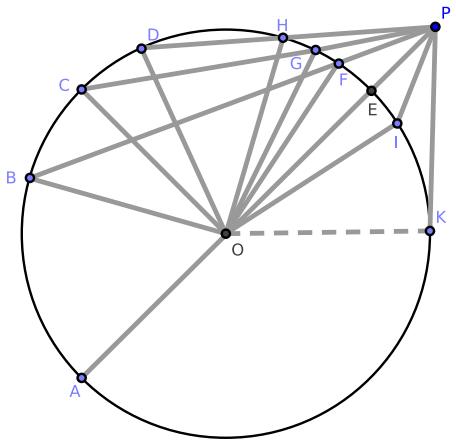


FIGURE 3.2.9. [3.8] $\bigcirc O$

1. The maximum segment passes through the center.

Construct $\overline{OB}$. Clearly, $\overline{OA} = \overline{OB}$. From this we obtain $\overline{AP} = \overline{OA} + \overline{OP} = \overline{OB} + \overline{OP}$. Consider $\triangle BOP$: $\overline{OB} + \overline{OP} > \overline{BP}$ [1.20]. Therefore, $\overline{AP} > \overline{BP}$.

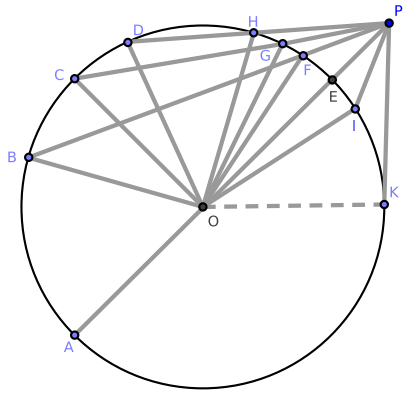


FIGURE 3.2.10. [3.8] $\bigcirc O$

2. The segments nearer to the segment through the center are greater in length than those which are farther.

Construct $\overline{OC}$ and $\overline{OD}$, and consider $\triangle BOP$ and $\triangle COP$: $\overline{OB} = \overline{OC}$, each share side $\overline{OP}$, and the angle $\angle BOP > \angle COP$. Therefore, $\overline{BP} > \overline{CP}$ [1.24]. Similarly, $\overline{CP} > \overline{DP}$, etc.

3. If segments are constructed to the convex circumference, the minimum segment is that which passes through the center when extended.

Construct $\overline{OF}$, and consider $\triangle OFP$: $\overline{OF} + \overline{FP} > \overline{OP} = \overline{OE} + \overline{EP}$ [1.20]. Since $\overline{OF} = \overline{OE}$, we have that $\overline{FP} > \overline{EP}$.

4. Of the other segments, that which is nearer to the minimum is smaller than one more farther out.

Construct $\overline{OG}$ and $\overline{OH}$, and consider $\triangle GOP$ and $\triangle FOP$: $\overline{GO} = \overline{FO}$, each shares side $\overline{OP}$, and the angle $\angle GOP > \angle FOP$. By [1.24], $\overline{GP} > \overline{FP}$. Similarly, $\overline{HP} > \overline{GP}$.

5. From the given point outside of the circle, there can be constructed two equal segments to the concave or the convex circumference, both of which make equal angles with the line passing through the center.

Construct the angles $\angle POI = \angle POF$ [1.23], and construct $\overline{IP}$. Consider: $\triangle IOP$ and $\triangle FOP$: $\overline{IO} = \overline{FO}$, each shares side $\overline{OP}$, and $\angle IOP = \angle POF$ by construction. By [1.4], $\overline{IP} = \overline{FP}$. Segments $\overline{IP}$ and $\overline{FP}$ fulfill the above requirements, completing this part of the proof.

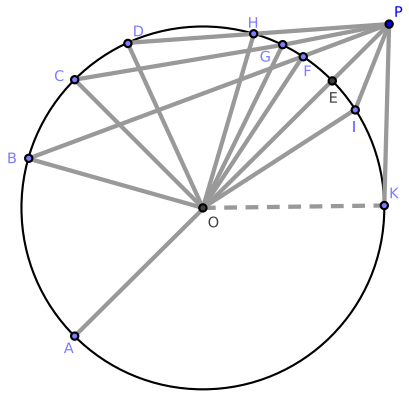


FIGURE 3.2.11. [3.8] $\bigcirc O$

6. Three or more equal segments cannot be constructed from the given point outside the circle to either circumference.

We claim that a third segment cannot is constructed from P equal to either of the segments $\overline{IP}$ and $\overline{FP}$. Suppose this were possible and let $\overline{PK} = \overline{PF}$. Then $\overline{PK} = \overline{PI}$, which contradicts claim 4. □

COROLLARY. *3.8.1. If $\overline{PI}$ is extended to meet the circle at L, then $\overline{PL} = \overline{PB}$.*

COROLLARY. *3.8.2. If two equal segments are constructed from P to either the convex or concave circumference, the diameter through P bisects the angle between them, and the segments intercepted by the circle are equal in length.*

COROLLARY. *3.8.3. If P is the common center of circles whose radii are segments constructed from P to the circumference of $\bigcirc O$, then:*

a) The circle whose radius is the minimum segment ($\overline{PE}$) has external contact with $\bigcirc O$ [Def. 3.4].

b) The circle whose radius is the maximum segment ($\overline{PA}$) has internal contact with $\bigcirc O$.

c) A circle having any of the remaining segments ($\overline{PF}$) as radius intersects $\bigcirc O$ at two points (F, I).

Exercises.

1. Prove [Cor. 3.8.1].

2. Prove [Cor. 3.8.2].

3. Prove [Cor. 3.8.3].

PROPOSITION 3.9. *THE CENTER OF A CIRCLE II.*

A point within a circle from which three or more equal segments can be constructed to the circumference is the center of that circle.

PROOF. Construct $\bigcirc ABC$ and equal segments $\overline{DA}$, $\overline{DB}$, and $\overline{DC}$. We claim that D is the center of $\bigcirc ABC$.

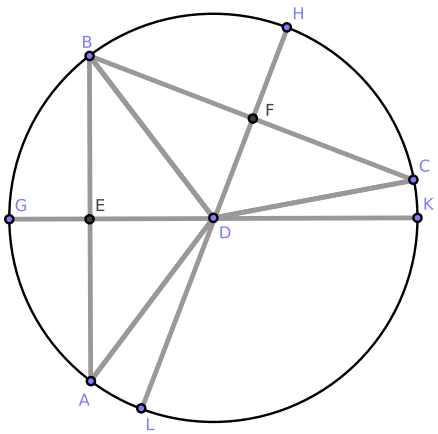


FIGURE 3.2.12. [3.9]

Construct $\overline{AB}$ and $\overline{BC}$ and bisect them at points E and F, respectively [1.10]. Then construct $\overline{GEDK}$ and $\overline{LDFH}$.

Consider $\triangle AED$ and $\triangle BED$: $\overline{ED}$ is a common side, $\overline{AE} = \overline{EB}$, and $\overline{DB} = \overline{DA}$ since each are radii of $\bigcirc ABC$. By [1.8], $\triangle AED \cong \triangle BED$, and so $\angle AED = \angle BED$; it follows that $\angle AED$ and $\angle BED$ are each right angles.

Since $\overline{GEDK} \perp \overline{AB}$ and $\overline{GEDK}$ bisects $\overline{AB}$, [3.1, Cor. 1] states that the center of $\bigcirc ABC$ is a point on $\overline{GEDK}$. Similarly, the center of $\bigcirc ABC$ is a point on $\overline{LDFH}$. Since $\overline{GEDK}$ and $\overline{LDFH}$ have no other point in common except for D, it follows that D is the center of $\bigcirc ABC$. □

Alternatively:

PROOF. Since $\overline{AD} = \overline{LD}$, the segment bisecting the angle $\angle ADL$ passes through the center [3.7, Cor. 1]. Similarly, the segment bisecting the angle $\angle BDA$ passes through the center. Hence, the point of intersection of these bisectors, D, is the center. □

PROPOSITION 3.10. *THE UNIQUENESS OF CIRCLES.*

If two circles have more than two points of their circumferences in common, they coincide.

PROOF. Construct $\bigcirc ABC$ and $\bigcirc DAB$ such that they have more than two points in common. We claim that $\bigcirc ABC$ and $\bigcirc DAB$ coincide.

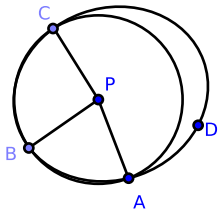


FIGURE 3.2.13. [3.10]

Suppose that $\bigcirc ABC$ and $\bigcirc DAB$ share three points in common (A, B, C) without coinciding. From P, the center of $\bigcirc ABC$, construct $\overline{PA}$, $\overline{PB}$, $\overline{PC}$; since each segment is a radius, it follows that $\overline{PA} = \overline{PB} = \overline{PC}$.

Since $\bigcirc DAB$ is a circle and P a point from which three equal lines $\overline{PA}$, $\overline{PB}$, $\overline{PC}$ can be constructed to its circumference, P must also be the center of $\bigcirc DAB$ by [3.9]; therefore $\bigcirc ABC$ and $\bigcirc DAB$ coincide, a contradiction.

This proves our claim. □

COROLLARY. *3.10.1. Two circles which do not coincide do not have more than two points common.*

Note: Similarly to [3.10, Cor. 1], two lines which do not coincide cannot have more than one point common.

Exercise.

1. Prove [Cor. 3.10.1].

PROPOSITION 3.11. *SEGMENTS CONTAINING CENTERS OF CIRCLES.*

If one circle touches another circle internally at one point, then the segment joining the centers of the two circles and terminating on the circumference must have that point of intersection as an endpoint.

PROOF. Construct $\bigcirc O$ and $\bigcirc H$ such that $\bigcirc H$ touches $\bigcirc O$ internally at P. We claim that any segment which contains O and H must also contain P (this segment may be extended if necessary).

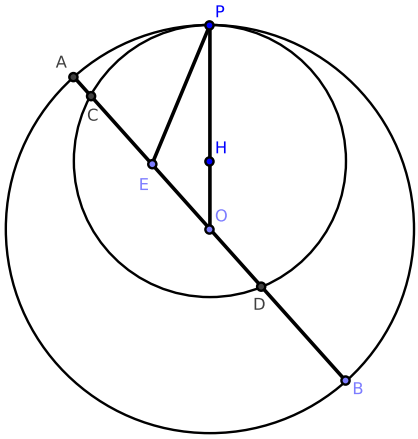


FIGURE 3.2.14. [3.11]

Suppose instead that the center of the internal circle is E, and construct $\overline{OP}$ and $\overline{EP}$.

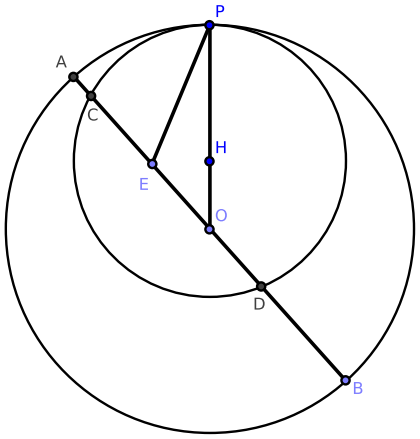
Extend $\overline{OE}$ to intersect $\bigcirc H$ at C and D and intersect $\bigcirc O$ at A and B. Since E is a point on the diameter of $\bigcirc O$ between O and A, $\overline{EA} < \overline{EP}$ [3.7].

But $\overline{EP} = \overline{EC}$ by hypothesis since they are radii of $\bigcirc H$. Hence $\overline{EA} < \overline{EC}$ where $\overline{EA} = \overline{EC} \oplus \overline{CA}$, a contradiction. Consequently, the center of the internal circle must lie on $\overline{OP}$.

The completes our claim. □

Alternatively:

PROOF. Since $\overline{EP}$ is a segment constructed from a point within the circle $\bigcirc O$ to the circumference of $\bigcirc O$ but not forming part of the diameter through E, the circle whose center is E with radius $\overline{EP}$ cuts $\bigcirc H$ at P [3.7, Cor. 2] and also touches it at P by hypothesis, a contradiction. A similar argument holds for all points not on $\overline{OP}$. Hence, the center of $\bigcirc O$ must be on $\overline{OP}$. □



PROPOSITION 3.12. *INTERSECTING CIRCLES I.*

If two circles intersect externally at one point, then the segment joining their centers contains that point of intersection.

PROOF. Construct $\bigcirc PCF$ and $\bigcirc PDE$ which intersect externally at point P. Construct $\overline{APE}$; we claim that the center of $\bigcirc PDE$ is a point on $\overline{APE}$.

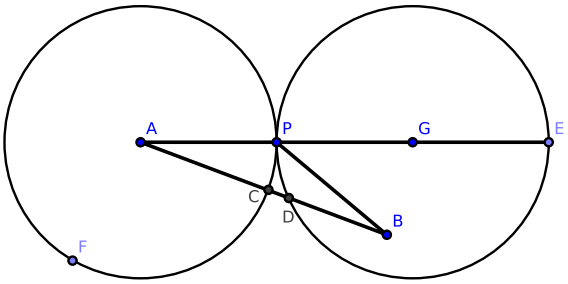


FIGURE 3.2.15. [3.12]

Suppose instead that the center of $\bigcirc PDE$ is a point not on $\overline{APE}$: wlog, suppose B is the center.

Construct $\overline{AB}$, intersecting $\bigcirc PDE$ at D and $\bigcirc PCF$ at C. Also construct $\overline{BP}$. By our hypothesis, $\overline{AP} = \overline{AC}$ and $\overline{BP} = \overline{BD}$. Hence

$$\overline{AP} + \overline{BP} = \overline{AC} + \overline{DB}$$

Also notice that $\overline{AB} = \overline{AC} \oplus \overline{CD} \oplus \overline{DB}$ where $\overline{CD} > 0$; it follows that $\overline{AB} > \overline{AC} + \overline{DB}$. By the above equality, $\overline{AB} > \overline{AP} + \overline{BP}$.

Consider $\triangle APB$: we have that one side of $\triangle APB$ is greater than the sum of the other two, which contradicts [1.20].

Since this contradiction occurs regardless of our choice of point within $\bigcirc PDE$ (except for G), the center of $\bigcirc PDF$ lies on $\overline{APE}$ at G. This proves our claim. $\square$

Alternatively:

PROOF. Suppose that the center of $\bigcirc PDE$ lies on $\overline{BP}$. Since $\overline{BP}$ is a segment constructed from a point outside of the circle $\bigcirc PCF$ to its circumference which does not pass through the center when it is extended, the circle whose center is B with radius $\overline{BP}$ must cut the circle $\bigcirc PCF$ at P [3.8, Cor. 3].

However, such a circle touches $\bigcirc PCF$ at P by hypothesis, a contradiction. Since $\overline{BP}$ was chosen as any segment other than $\overline{PE}$, the center of $\bigcirc PDE$ must lie on $\overline{PE}$. □

REMARK. [3.11] and [3.12] may both be included in one theorem: "If two circles touch each other at any point, the centers and that point are collinear." This is a limiting case of the theorem given in [3.3, Cor. 4]: "The line joining the centers of two intersecting circles bisects the common chord perpendicularly."

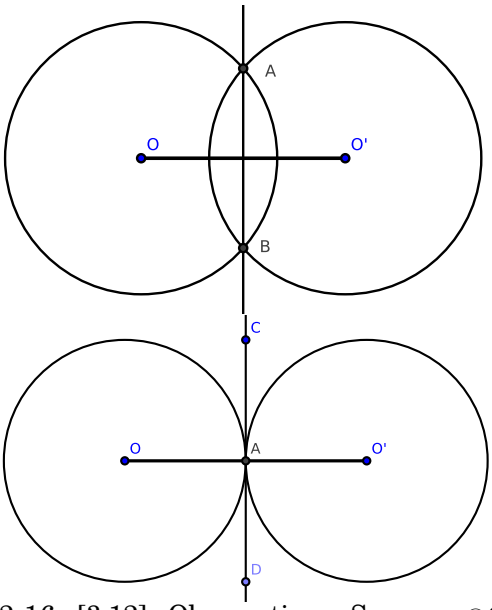


FIGURE 3.2.16. [3.12], Observation. Suppose $\bigcirc O$ and $\bigcirc O'$ have two points of intersection, A and B. Suppose further that A remains fixed while the second circle moves so that the point B ultimately coincides with A. Since the segment $\overline{OO'}$ always bisects $\overline{AB}$, we see that $\overline{OO'}$ intersects A. In consequence of this motion, the common chord $\overline{CD}$ becomes the tangent to each circle at A.

COROLLARY. *3.12.1. If two circles touch each other, their point of intersection is the union of two points of intersection. When counting the number of points at which two circles intersect, we may for purposes of calculation consider this point of intersection as two points.*

COROLLARY. *3.12.2. If two circles touch each other at any point, they cannot have any other common point.*

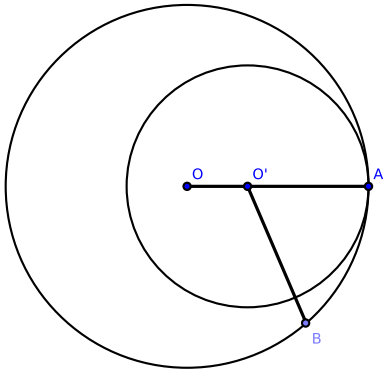


FIGURE 3.2.17. [3.2, Cor. 2]

For, since two circles cannot have more than two points common [3.10] and their point of intersection is equivalent to two points for purposes of calculation, circles that touch cannot have any other point common. The following is a formal proof of this Corollary:

Construct $\bigcirc O$ *and* $\bigcirc O'$ *where* A *is the point of intersection, and let* O' *lie between* O *and* A*. Take any other point* B *in the circumference of* $\bigcirc O$*, and construct* $\overline{O'B}$*. By [3.7],* $\overline{O'B} > \overline{O'A}$*. Therefore,* B *is outside the circumference of the inner circle. Hence,* B *cannot be common to both circles. Since point* B *was chosen arbitrarily, the circles cannot have any other common point except for* A*.*

PROPOSITION 3.13. *INTERSECTING CIRCLES II.*

Two circles cannot touch each other at two points either internally or externally.

PROOF. We divide the proof into its internal and external cases:

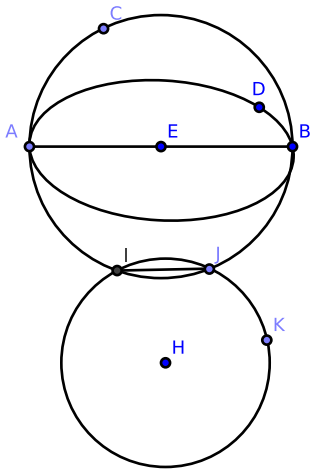


FIGURE 3.2.18. [3.13]

Internal case: suppose two distinct circles $\bigcirc ACB$ and $\bigcirc ADB$ touch internally at two points A and B. Since the two circles touch at A, the segment joining their centers passes through A [3.11]. Similarly, the segment joining their centers passes through B. Hence, the centers of these circles and the points A and B are on $\overline{AB}$, and so $\overline{AB}$ is a diameter of each circle. Bisect $\overline{AB}$ at E: clearly, E is the center of each circle, i.e., the circles are concentric. This contradicts [3.5], and thus $\bigcirc ACB$ and $\bigcirc ADB$ do not touch internally at two points.

External case: if two circles $\bigcirc E$ and $\bigcirc H$ touch externally at points I and J, then by [3.12] $\overline{EH}$ contains the points I and J, which contradicts the construction of Fig. 3.2.18. □

An alternative proof to the internal case:

PROOF. Construct a line bisecting $\overline{AB}$ perpendicularly. By [3.1, Cor. 1], this line passes through the center of each circle, and by [3.11] and [3.12] must

pass through each point of intersection, a contradiction. Hence, two circles cannot touch each other at two points. □

REMARK. This proposition is an immediate inference from [3.12, Cor. 1] that if a point of intersection counts for two intersections, then two contacts would be equivalent to four intersections; but there cannot be more than two intersections [3.10]. It also follows from [3.12, Cor. 2] that if two circles touch each other at point A, they cannot have any other point in common; hence, they cannot touch again at B.

Exercises.

1. If a circle with a non-fixed center touches two fixed circles externally, the difference between the distances of its center from the centers of the fixed circles is equal to the difference or the sum of their radii, according to whether the contacts are of the same or of opposite type [Def. 3.4].

2. If a circle with a non-fixed center is touched by one of two fixed circles internally and touches the other fixed circle either externally or internally, the sum of the distances from its center to the centers of the fixed circles is equal to the sum or the difference of their radii, according to whether the contact with the second circle is internal or external.

3. Suppose two circles touch externally. If through the point of intersection any secant is constructed cutting the circles again at two points, the radii constructed to these points are parallel. [See the final chapter for a solution.]

4. Suppose two circles touch externally. If two diameters in these circles are parallel, the line from the point of intersection to the endpoint of one diameter passes through the endpoint of the other. [See the final chapter for a solution.]

PROPOSITION 3.14. *EQUALITY OF CHORD LENGTHS.*

Chords in a circle are equal in length if and only if they are equally distant from the center.

PROOF. Construct $\bigcirc O$ with chords $\overline{AB}$ and $\overline{CD}$. We claim that:

(1) If $\overline{AB} = \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are equally distant from the center.

(2) If $\overline{AB}$ and $\overline{CD}$ are equally distant from the center, then $\overline{AB} = \overline{CD}$.

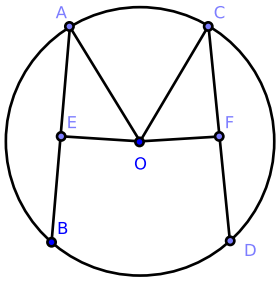


FIGURE 3.2.19. [3.14]

Claim 1: let $\overline{AB} = \overline{CD}$, and construct $\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$. We claim that $\overline{EO} = \overline{FO}$.

Construct $\overline{AO}$ and $\overline{CO}$. Because $\overline{AB}$ is a chord in $\bigcirc O$ and $\overline{OE}$ is a segment constructed from the center to E, $\overline{OE}$ bisects $\overline{AB}$ [3.3]; or $\overline{AE} = \overline{EB}$. Similarly, $\overline{CF} = \overline{FD}$. Since $\overline{AB} = \overline{CD}$ by hypothesis, $\overline{AE} = \overline{CF}$.

Because $\angle OEF$ is a right angle, $(\overline{AO})^2 = (\overline{AE})^2 + (\overline{EO})^2$ by [1.47]. Similarly, $(\overline{CO})^2 = (\overline{CF})^2 + (\overline{FO})^2$. Since $(\overline{AO})^2 = (\overline{CO})^2$ and $(\overline{AE})^2 = (\overline{CF})^2$, we have $(\overline{EO})^2 = (\overline{FO})^2$, and so $\overline{EO} = \overline{FO}$. This proves our claim.

Claim 2: let $\overline{EO} = \overline{FO}$, and construct $\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$. We claim that $\overline{AB} = \overline{CD}$.

Construct $\overline{AO}$ and $\overline{CO}$. By [1.47] and similarly to the above proof, $(\overline{AE})^2 + (\overline{EO})^2 = (\overline{CF})^2 + (\overline{FO})^2$ where $(\overline{EO})^2 = (\overline{FO})^2$ due to our hypothesis. Hence $(\overline{AE})^2 = (\overline{CF})^2$, and so $\overline{AE} = \overline{CF}$. But $\overline{AB} = 2 \cdot \overline{AE}$ and $\overline{CD} = 2 \cdot \overline{CF}$ by [3.3], and so $\overline{AB} = \overline{CD}$. □

Exercises.

1. If a chord of given length slides around a fixed circle, then:

   (a) the locus of its midpoint is a circle;

(b) the locus of any point fixed on the chord is a circle.

PROPOSITION 3.15. *INEQUALITY OF CHORD LENGTHS.*

The diameter is the longest chord in a circle, and a chord is nearer to the center of a circle than another chord if and only if it is the longer of the two chords.

PROOF. Construct $\bigcirc O$ with diameter $\overline{AB}$ and chords $\overline{CD}$ and $\overline{EF}$ such that $\overline{CD}$ is nearer to O that $\overline{EF}$. We claim that:

(1) $\overline{AB}$ is the longest chord in a circle;

(2) $\overline{CD} > \overline{EF}$;

(3) Longer chords are nearer to the center than shorter chords.

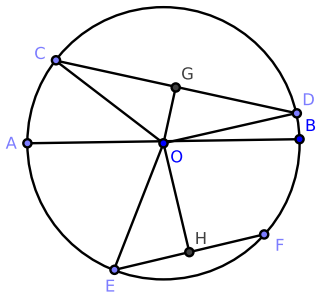


FIGURE 3.2.20. [3.15]

(1) We claim that $\overline{AB}$ is the longest chord in a circle.

Construct $\overline{OC}$, $\overline{OD}$, $\overline{OE}$ as well as $\overline{OG} \perp \overline{CD}$ and $\overline{OH} \perp \overline{EF}$. Notice that $\overline{AB} = \overline{OA} + \overline{OB} = \overline{OC} + \overline{OD}$. Consider $\triangle CDO$: $\overline{OC} + \overline{OD} > \overline{CD}$ by [1.20]. Therefore, $\overline{AB} > \overline{CD}$. Since the choice of $\overline{CD}$ was arbitrary, our claim is proven.

(2) We claim that $\overline{CD} > \overline{EF}$.

Since $\overline{CD}$ is nearer to O than $\overline{EF}$ by hypothesis, it follows that $\overline{OG} < \overline{OH}$ [3.14]. Since $\triangle OGC$ and $\triangle OHE$ are right triangles, we have that $(\overline{OC})^2 = (\overline{OG})^2 + (\overline{GC})^2$ and $(\overline{OE})^2 = (\overline{OH})^2 + (\overline{HE})^2$. Since $\overline{OC} = \overline{OE}$, $(\overline{OG})^2 + (\overline{GC})^2 = (\overline{OH})^2 + (\overline{HE})^2$. But $(\overline{OG})^2 < (\overline{OH})^2$, and so $(\overline{GC})^2 > (\overline{HE})^2$. It follows that $\overline{GC} > \overline{HE}$. Since $\overline{CD} = 2 \cdot \overline{GC}$ and $\overline{EF} = 2 \cdot \overline{HE}$ by [3.3], it follows that $\overline{CD} > \overline{EF}$.

(3) Longer chords are nearer to the center than shorter chords.

Suppose that $\overline{CD} > \overline{EF}$. We claim that $\overline{OG} < \overline{OH}$.

As before, we have that $(\overline{OG})^2 + (\overline{GC})^2 = (\overline{OH})^2 + (\overline{HE})^2$. Due to our hypothesis, $(\overline{GC})^2 > (\overline{HE})^2$. Therefore $(\overline{OG})^2 < (\overline{OH})^2$, and so $\overline{OG} < \overline{OH}$. □

Exercises.

1. The shortest chord which can be constructed through a given point within a circle is the perpendicular to the diameter which passes through that point.

2. Through a given point, within or outside of a given circle, construct a chord of length equal to that of a given chord.

3. Through one of the points of intersection of two circles, construct a secant

   (a) where the sum of its segments intercepted by the circles is a maximum;

   (b) which is of any length less than that of the maximum.

4. Suppose that circles touch each other externally at A, B, C and that the chords $\overline{AB}$, $\overline{AC}$ of two of them are extended to meet the third again in the points D and E. Prove that $\overline{DE}$ is a diameter of the third circle and is parallel to the segment joining the centers of the others.

PROPOSITION 3.16. *THE PERPENDICULAR TO A DIAMETER OF A CIRCLE.*

The perpendicular to a diameter of a circle intersects the circumference at one and only one point, and any other segment through the diameter's endpoint intersects the circle at two points.

PROOF. Construct $\bigcirc C$ with points A, B, and H on its circumference where $\overline{AB}$ is a diameter of $\bigcirc C$. Also construct $\overline{BH}$ and $\overleftrightarrow{BI} \perp \overline{AB}$ where $\overleftrightarrow{BI}$ intersects $\bigcirc C$ at B. We claim that:

(1) $\overleftrightarrow{BI}$ touches $\bigcirc C$ at B only;

(2) $\overline{BH}$ cuts $\bigcirc C$.

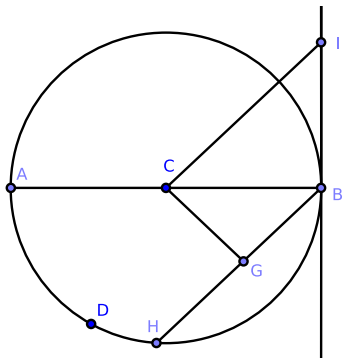


FIGURE 3.2.21. [3.16]

Claim 1: $\overleftrightarrow{BI}$ touches $\bigcirc C$ at B only.

Let I be an arbitrary point on $\overleftrightarrow{BI}$ and construct the segment $\overline{CI}$. Because $\angle CBI$ is a right angle, $(\overline{CI})^2 = (\overline{CB})^2 + (\overline{BI})^2$ by [1.47]. It follows that $(\overline{CI})^2 > (\overline{CB})^2$, and so $\overline{CI} > \overline{CB}$. By [3.2], I lies outside of $\bigcirc C$. Similarly, every other point on $\overleftrightarrow{BI}$ except B lies outside of the $\bigcirc C$. Since $\overleftrightarrow{BI}$ intersects $\bigcirc C$ at B and does not cut it, $\overleftrightarrow{BI}$ touches $\bigcirc C$ at B only.

Claim 2: $\overline{BH}$ cuts $\bigcirc C$.

Construct $\overline{CG} \perp \overline{BH}$. It follows that $(\overline{BC})^2 = (\overline{CG})^2 + (\overline{GB})^2$. Therefore $(\overline{BC})^2 > (\overline{CG})^2$, and so $\overline{BC} > \overline{CG}$. By [3.2], G must lie within $\bigcirc C$, and consequently if $\overline{BG}$ is extended it must also intersect $\bigcirc C$ at H and therefore cut it.

The proves our claim. □

Exercises.

1. If two circles are concentric, all chords of the greater circle which touch the lesser circle are equal in length. [See the final chapter for a solution.]

2. Construct a parallel to a given line which touches a given circle.

3. Construct a perpendicular to a given line which touches a given circle.

4. Construct a circle having its center at a given point

   (a) and touches a given line;

   (b) and touches a given circle.

How many solutions exist in this case?

5. Construct a circle of given radius that touches two given lines. How many solutions exist?

6. Find the locus of the centers of a system of circles touching two given lines.

7. Construct a circle of given radius that touches a given circle and a given line or that touches two given circles.

PROPOSITION 3.17. *TANGENTS ON CIRCLES I.*

It is possible to construct a tangent of a given circle from a given point outside of the circle.

PROOF. Construct $\bigcirc O$ and P such that P is outside of $\bigcirc O$. We wish to construct tangent $\overline{BP}$ to $\bigcirc O$.

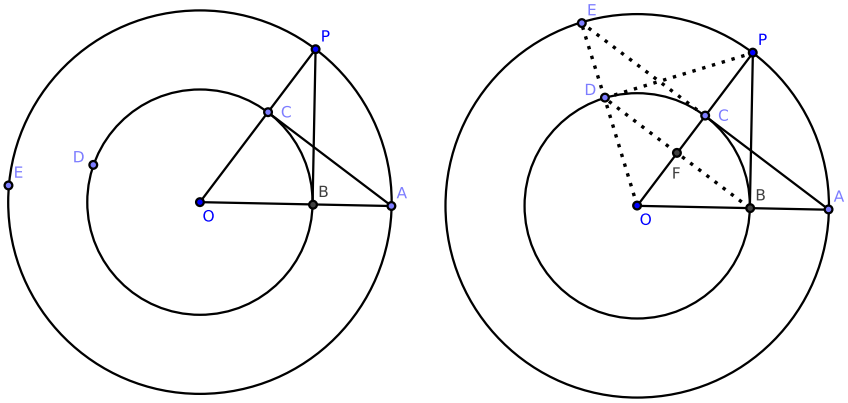


FIGURE 3.2.22. [3.17] (α), (β)

Construct radius $\overline{OC}$ and extend $\overline{OC}$ to $\overline{OP}$. With O as center and $\overline{OP}$ as radius, construct the circle $\bigcirc O_1$. Also construct $\overline{CA} \perp \overline{OP}$. Construct $\overline{OA}$, intersecting $\bigcirc O$ at B, and construct $\overline{BP}$. We claim that $\overline{BP}$ is the required tangent to $\bigcirc O$.

Since O is the center of $\bigcirc O$ and $\bigcirc O_1$, we have that $\overline{OA} = \overline{OP}$ and $\overline{OC} = \overline{OB}$. Consider $\triangle AOC$ and $\triangle POB$: $\overline{OA} = \overline{OP}$, $\overline{OC} = \overline{OB}$, and each shares $\angle BOC = \angle AOC = \angle POB$. By [1.4], $\triangle AOC \cong \triangle POB$, and so $\angle OCA = \angle OBP$.

But $\angle OCA$ is a right angle by construction; therefore $\angle OBP$ is a right angle, and by [3.16], $\overline{BP}$ touches the circle $\bigcirc O$ at B. By definition $\overline{BP}$ is a tangent of $\bigcirc O$ at point B, which proves our claim. □

COROLLARY. *3.17.1. If $\overline{AC}$ is extended to $\overline{AE}$ and $\overline{OE}$ is constructed, then $\bigcirc O$ is cut at D. Construct $\overline{DP}$; $\overline{DP}$ is a second tangent of $\bigcirc O$ at P.*

Exercises.

1. Prove that tangents $\overline{PB}$ and $\overline{PD}$ in [3.17] are equal in length because the square of each is equal to the square of $\overline{OP}$ minus the square of the radius.

2. If a quadrilateral is circumscribed to a circle, the sum of the lengths of one pair of opposite sides is equal to the sum of the lengths of the other pair.

3. If a parallelogram is circumscribed to a circle, it must be a rhombus, and so its diagonals intersect at the center. [See the final chapter for a solution.]

4. If $\overline{BD}$ is constructed and $\overline{OP}$ is intersected at F, then $\overline{OP} \perp \overline{BD}$.

5. The locus of the intersection of two equal tangents to two circles is a segment (called the *radical axis* of the two circles).

6. Find a point such that tangents from it to three given circles is equal. (This point is called the *radical center* of the three circles.)

7. Prove that the rectangle $OF \cdot OP$ is equal in area to the square of the radius of $\bigcirc O$. (Note: we are locating the inverse points with respect to $\bigcirc O$. See the definition below.)

Definition. 16. Suppose we have two points F and P such that when the area of the rectangle $OF \cdot OP$ is equal to the area of the square of the radius of that circle, then F and P are called *inverse points* with respect to the circle.

9. The intercept made on a variable tangent by two fixed tangents stands opposite a constant angle at the center.

10. Construct a common tangent to two circles. Demonstrate how to construct a segment cutting two circles so that the intercepted chords are of given lengths.

11. Prove [Cor. 3.17.1].

PROPOSITION 3.18. *TANGENTS ON CIRCLES II.*

If a line touches a circle, the segment from the center of the circle to the point of intersection with the line is perpendicular to the line.

PROOF. Construct $\bigcirc O$ with point C on its circumference, and also construct $\overleftrightarrow{CD}$. We claim that if $\overleftrightarrow{CD}$ touches $\bigcirc O$, then $\overline{OC} \perp \overleftrightarrow{CD}$.

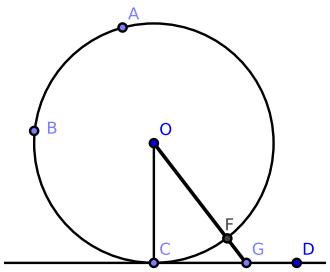


FIGURE 3.2.23. [3.18]

Instead, suppose that another segment $\overline{OG}$ is constructed from the center such that $\overline{OG} \perp \overleftrightarrow{CD}$ where $\overline{OG}$ cuts the circle at F. Because the angle $\angle OGC$ is right by hypothesis, the angle $\angle OCG$ must be acute [1.17]. By [1.19], $\overline{OC} > \overline{OG}$. But $\overline{OG} = \overline{OF} \oplus \overline{FG}$ and $\overline{OC} = \overline{OF}$, and so $\overline{OC} < \overline{OG}$, a contradiction. Hence $\overline{OC} \perp \overleftrightarrow{CD}$. □

Alternatively:

PROOF. Since the perpendicular must be the shortest segment from O to $\overleftrightarrow{CD}$ and $\overline{OC}$ is evidently the shortest line, it follows that $\overline{OC} \perp \overleftrightarrow{CD}$. □

PROPOSITION 3.19. *TANGENTS ON CIRCLES III.*

If a line is a tangent to a circle, then the perpendicular constructed from its point of intersection passes through the center of the circle.

PROOF. Suppose $\overleftrightarrow{AB}$ is tangent to $\bigcirc CDA$. We claim that if $\overleftrightarrow{AB} \perp \overline{AC}$, then $\overline{AC}$ contains the center of $\bigcirc CDA$.

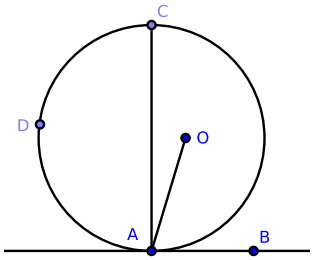


FIGURE 3.2.24. [3.19]

Suppose otherwise: let O be the center of $\bigcirc CDA$ and construct $\overline{AO}$. Notice that $\angle OAC > 0$. Because $\overleftrightarrow{AB}$ touches $\bigcirc CDA$ and $\overline{OA}$ is constructed from the center to the point of intersection, $\overline{OA} \perp \overleftrightarrow{AB}$ by [3.18]. Since $\overleftrightarrow{AB} \perp \overline{AC}$ by hypothesis, $\angle OAB$ and $\angle CAB$ are right angles.

It follows that $\angle CAB = \angle OAB$ and $\angle CAB = \angle OAB + \angle OAC$; hence $\angle OAC = 0$ and $\angle OAC > 0$, a contradiction. Therefore, the center lies on $\overline{AC}$. □

COROLLARY. *3.19.1. If a number of circles touch the same line at the same point, the locus of their centers is the perpendicular to the line at the point.*

COROLLARY. *3.19.2. Suppose we have a circle and any two of the following properties:*

(a) a tangent to a circumference;

(b) a segment, ray, or straight line constructed from the center of the circle to the point of intersection;

(c) right angles at the point of intersection.

Then by [3.16], [3.18], [3.19], and the Rule of Symmetry, the remaining property follows. If we have (a) and (c), then it may be necessary to extend a given segment or a ray to the center of the circle: these are limiting cases of [3.1, Cor. 1] and [3.3].

PROPOSITION 3.20. *ANGLES AT THE CENTER OF A CIRCLE AND ON THE CIRCUMFERENCE.*

The angle at the center of a circle is double the angle at the circumference when each stands on the same arc of the circumference.

PROOF. Construct $\bigcirc E$ with $\overline{EB}$ as radius, and construct $\angle BEC$ and $\angle BAC$ where A and C are points on the circumference of $\bigcirc E$. We claim that:

(1)$\angle BEC = 2 \cdot \angle BAC$

(2) $\angle BEC = 2 \cdot \angle BDC$

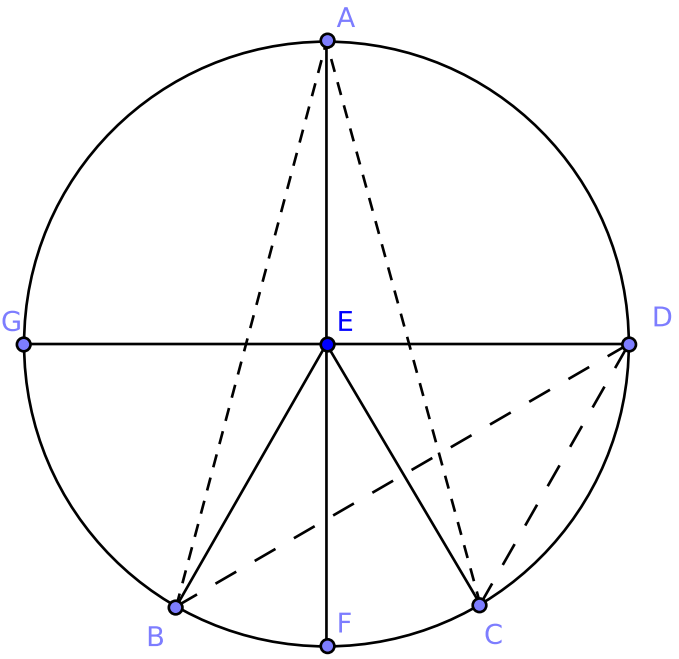


FIGURE 3.2.25. [3.20]

Construct $\overline{AEF}$, and consider $\triangle EAB$: since $\overline{EA} = \overline{EB}$, by [1.6] $\angle EAB = \angle EBA$ and therefore $\angle EAB + \angle EBA = 2 \cdot \angle EAB$.

Since $\angle BEF = \angle EAB + \angle EBA$ by [1.32], $\angle BEF = 2 \cdot \angle EAB$. Similarly in $\triangle EAC$, $\angle FEC = 2 \cdot \angle EAC$. It follows that $\angle BEC = 2 \cdot \angle BAC$ (claim 1).

Now construct $\overline{GD}$, $\overline{BD}$, and $\overline{CD}$. By an argument similar to the above, we can prove that $\angle GEC = 2 \cdot \angle EDC$ and $\angle GEB = 2 \cdot \angle EDB$. Since $\angle BEC =$

$\angle GEC - \angle GEB$, we have that

$$\angle BEC = 2 \cdot (\angle EDC - \angle EDB) = 2 \cdot \angle BDC$$

(claim 2). This completes the proof. □

COROLLARY. *3.20.1. The angle in a semicircle is a right angle.*

PROPOSITION 3.21. *ANGLES ON CHORDS.*

In a circle, angles standing on the same arc are equal in measure to each other.

PROOF. Construct $\bigcirc F$, and also construct $\angle BAD$ and $\angle BED$ on the same arc BCD. We claim that $\angle BAD = \angle BED$.

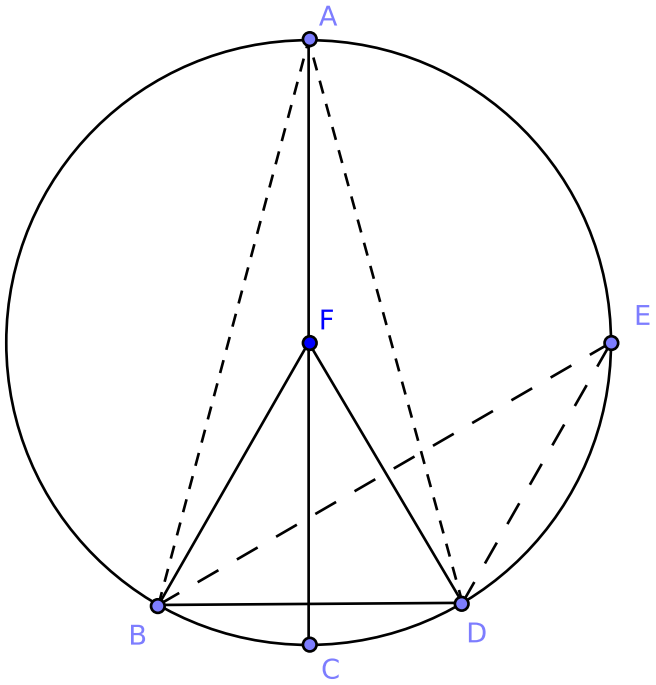


FIGURE 3.2.26. [3.21]

Construct $\overline{BF}$ and $\overline{FD}$ [3.1]. Since $\angle BFD$ is at the center of $\bigcirc F$ and $\angle BAD$ is at the circumference of $\bigcirc E$, and because they each have BCD on the circumference at their bases, by [3.20] $\angle BFD = 2 \cdot \angle BAD$.

Similarly, $\angle BFD = 2 \cdot \angle BED$; therefore, $\angle BAD = \angle BED$, which proves our claim. □

COROLLARY. *3.21.1. If two triangles $\triangle ACB$, $\triangle ADB$ stand on the same base $\overline{AB}$ and have equal vertical angles on the same side of it, then the four points A, C, D, B are concyclic.*

COROLLARY. *3.21.2. If A, B are two fixed points and if C varies its position in such a way that the angle $\angle ACB$ retains the same value throughout, the locus of C is a circle. (Or: given the base of a triangle and the vertical angle, the locus of the vertex is a circle).*

Exercises.

1. Given the base of a triangle and the vertical angle, find the locus

(a) of the intersection of its perpendiculars;

(b) of the intersection of the internal bisectors of its base angles;

(c) of the intersection of the external bisectors of the base angles;

(d) of the intersection of the external bisector of one base angle and the internal bisector of the other.

2. If the sum of the squares of two segments is given, prove that their sum is a maximum when the segments are equal in length.

3. Of all triangles having the same base and vertical angle, prove that the sum of the sides of an isosceles triangle is a maximum.

4. Of all triangles inscribed in a circle, the equilateral triangle has the maximum perimeter.

5. Of all concyclic figures having a given number of sides, the area is a maximum when the sides are equal.

6. Prove [Cor. 3.20.1].

7. Prove [Cor. 3.21.1].

8. Prove [Cor. 3.21.2].

PROPOSITION 3.22. *QUADRILATERALS INSCRIBED INSIDE CIRCLES.*

The sum of the opposite angles of a quadrilateral inscribed in a circle equals two right angles.

PROOF. Construct quadrilateral $ABCD$ inscribed in $\bigcirc E$. We claim that the sum of the opposite angles of $ABCD$ equals two right angles.

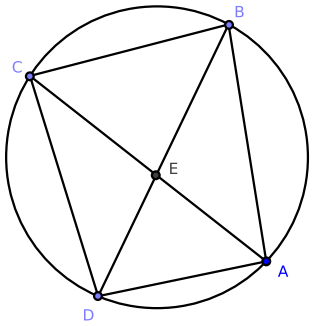


FIGURE 3.2.27. [3.22]

Let $\rho =$ two right angles, and construct diameters $\overline{AC}$ and $\overline{BD}$. Since $\angle ABD$ and $\angle ACD$ stand on arc AD, $\angle ABD = \angle ACD$ by [3.21]. Similarly, $\angle DBC = \angle DAC$ because they stand on arc DC. Hence

$$\begin{aligned} \angle ABC &= \angle ABD + \angle DBC \\ &= \angle ACD + \angle DAC \end{aligned}$$

From this, we obtain

$$\angle ABC + \angle CDA = \angle ACD + \angle DAC + \angle CDA$$

where the right-hand side of the equality is the sum of the interior angles of $\triangle ACD$. Since this sum equals ρ by [1.32], $\angle ABC + \angle CDA = \rho$.

Similarly, $\angle DAB + \angle BCD = \rho$ (*mutatis mutandis*), which proves our claim. □

COROLLARY. *3.22.1. If the sum of two opposite angles of a quadrilateral are equal to two right angles, then a circle may be inscribed about the quadrilateral.*

COROLLARY. *3.22.2. If a parallelogram is inscribed in a circle, then it is a rectangle.*

Exercises.

1. If the opposite angles of a quadrilateral are supplemental, it is cyclic.

2. A segment which makes equal angles with one pair of opposite sides of a cyclic quadrilateral makes equal angles with the remaining pair and with the diagonals.

3. If two opposite sides of a cyclic quadrilateral are extended to meet and a perpendicular falls on the bisector of the angle between them from the point of intersection of the diagonals, this perpendicular will bisect the angle between the diagonals.

4. If two pairs of opposite sides of a cyclic hexagon are respectively parallel to each other, the remaining pair of sides are also parallel.

5. If two circles intersect at the points A, B, and any two segments $\overline{ACD}$, $\overline{BFE}$ are constructed through A and B, cutting one of the circles in the points C, E and the other in the points D, F, then $CE \parallel DF$.

6. If equilateral triangles are constructed on the sides of any triangle, the segments joining the vertices of the original triangle to the opposite vertices of the equilateral triangles are concurrent.

7. In the same case as #7, prove that the centers of the circles constructed about the equilateral triangles form another equilateral triangle.

8. If a quadrilateral is constructed about a circle, the angles at the center standing opposite the opposite sides are supplemental.

9. If a tangent which varies in position meets two parallel tangents, it stands opposite a right angle at the center.

10. If a hexagon is circumscribed about a circle, the sum of the angles standing opposite the center from any three alternate sides is equal to two right angles.

11. Prove [Cor. 3.22.1].

12. After completing #11, rewrite the results of [3.22] and [Cor. 3.22.1] into one proposition.

13. Prove [Cor. 3.22.2].

PROPOSITION 3.23. *UNIQUENESS OF ARCS.*

Two similar and unequal arcs cannot be constructed on the same side of the same chord.

PROOF. Construct $\overline{AB}$. Suppose instead that two similar and unequal arcs ACB and ADB are constructed on the same side of $\overline{AB}$. Construct $\overline{ADC}$, $\overline{CB}$, and $\overline{DB}$.

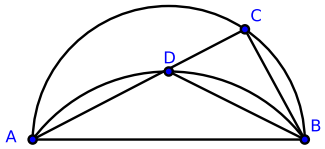


FIGURE 3.2.28. [3.23]

Since arc ACB is similar to arc ADB, $\angle ADB = \angle ACB$ by [Def. 3.10]; this contradicts [1.16]. This proves our claim. □

PROPOSITION 3.24. *EQUALITY OF SIMILAR ARCS.*

Similar arcs standing on equal chords are equal in length.

PROOF. Construct $\overline{AB} = \overline{CD}$ and arcs AEB and CFD such that $AEB \sim CFD$. We claim $AEB = CFD$.

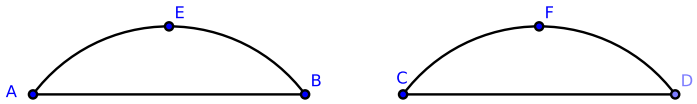


FIGURE 3.2.29. [3.24]

Since $\overline{AB} = \overline{CD}$, if $\overline{AB}$ is applied to $\overline{CD}$ such that the point A coincides with C and the point B coincides with D, then the chord $\overline{AB}$ coincides with $\overline{CD}$. Because $AEB \sim CFD$, they must coincide at every point [3.23]. This proves our claim. □

COROLLARY. *3.24.1. Since the chords are equal in length, they are congruent; therefore the arcs, being similar, are also congruent.*

PROPOSITION 3.25. *CONSTRUCTION OF A CIRCLE FROM AN ARC.*

Given an arc of a circle, it is possible to construct the circle to which the arc belongs.

PROOF. Given an arc (ABC) of $\bigcirc F$, we wish to construct $\bigcirc F$.

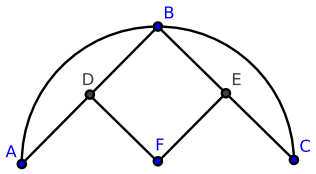


FIGURE 3.2.30. [3.25]

Take any three points A, B, C on arc ABC. Construct $\overline{AB}$, $\overline{BC}$. Bisect $\overline{AB}$ at D and $\overline{BC}$ at E. Construct $\overline{DF} \perp \overline{AB}$ and $\overline{EF} \perp \overline{BC}$. We claim that F, the intersection of $\overline{DF}$ and $\overline{EF}$, is the center of the required circle.

Because $\overline{DF}$ bisects and is perpendicular to $\overline{AB}$, the center of the circle of which ABC is an arc lies on $\overline{DF}$ [3.1, Cor. 1]. Similarly, the center of the circle of which ABC is an arc lies on $\overline{EF}$.

Since F is the intersection of $\overline{DF}$ and $\overline{EF}$, our claim is proven. □

PROPOSITION 3.26. *ANGLES AND ARCS I.*

In equal circles, equal angles at the centers or on the circumferences stand upon arcs of equal length.

PROOF. Construct $\bigcirc G$ and $\bigcirc H$ to be equal in all respects and with equal angles at the centers: $\angle BGC$ in $\bigcirc G$ and $\angle EHF$ in $\bigcirc H$. Also construct equal angles $\angle BAC$ in $\bigcirc G$ where A is on the circumference and $\angle EDF$ in $\bigcirc H$ where D is on the circumference.

We claim that arc BKC =arc ELF.

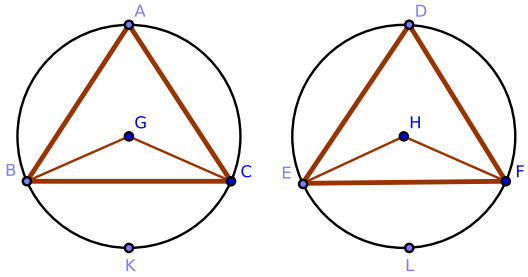


FIGURE 3.2.31. [3.26]

Construct $\overline{BC}$ and $\overline{EF}$ and consider $\triangle BGC$ and $\triangle EHF$: $\overline{BG} = \overline{EH}$, $\overline{GC} = \overline{HF}$, and $\angle BGC = \angle EHF$. By [1.4], $\triangle BGC \cong \triangle EHF$, and so $\overline{BC} = \overline{EF}$.

By [3.20], $\angle BAC = \frac{1}{2} \cdot \angle BGC$ and $\angle EDF = \frac{1}{2} \cdot \angle EHF$. Since $\angle BGC = \angle EHF$ by hypothesis, $\angle BAC = \angle EDF$. By [Def. 3.10], arc BAC ~arc EDF. And by [3.24], arc BAC =arc EDF.

Since $\bigcirc G$ and $\bigcirc H$ are equal in all respects by hypothesis, we also have that arc BKC =arc ELF. This proves our claim. □

COROLLARY. *3.26.1. If the opposite angles of a cyclic quadrilateral are equal, one of its diagonals must be a diameter of the circumscribed circle.*

COROLLARY. *3.26.2. Parallel chords in a circle intercept equal arcs.*

COROLLARY. *3.26.3. If two chords intersect at any point within a circle, the sum of the opposite arcs which they intercept is equal to the arc which parallel chords intersecting on the circumference intercept. If two chords intersect at any point outside a circle, the difference of the arcs they intercept is equal to the arc which parallel chords intersecting on the circumference intercept.*

COROLLARY. *3.26.4. If two chords intersect at right angles, the sum of the opposite arcs which they intercept on the circle is a semicircle.*

Exercises.

1. Prove [Cor. 3.26.1].
2. Prove [Cor. 3.26.2].
3. Prove [Cor. 3.26.3].
4. Prove [Cor. 3.26.4].

PROPOSITION 3.27. *ANGLES AND ARCS II.*

In equal circles, angles at the centers or at the circumferences which stand on equal arcs are equal in measure.

PROOF. Construct $\bigcirc G$ and $\bigcirc H$ to be equal in all respects and also construct angles at the centers ($\angle AOB$, $\angle DHE$) and

at the circumferences ($\angle ACB$, $\angle DFE$) of $\bigcirc G$ and $\bigcirc H$ which stand on equal arcs (AGB, DKE). We claim that $\angle ACB = \angle DFE$.

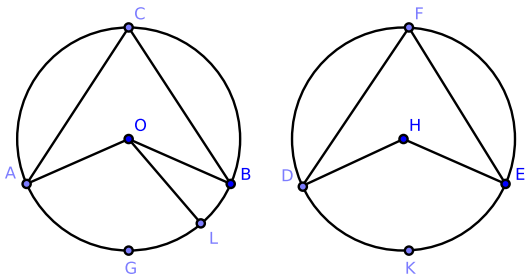


FIGURE 3.2.32. [3.27]

Consider the angles at the centers ($\angle AOB$, $\angle DHE$). Suppose that $\angle AOB > \angle DHE$ and that $\angle AOL = \angle DHE$. Since the circles are equal in all respects, arc $AGL =$ arc DKE [3.26]. Notice that $LB > 0$.

But arc $AGB =$arc DKE by hypothesis. Hence arc $AGB =$arc AGL and arc $AGB =$arc $AGL \oplus$arc LB where arc $LB = 0$, a contradiction. A corresponding contradiction follows if we assume that $\angle AOB < \angle DHE$. Therefore, $\angle AOB = \angle DHE$.

Now consider the angles at the circumference. Since $2 \cdot \angle ACB = \angle AOB$ and $2 \cdot \angle DFE = \angle DHE$ by [3.20], $\angle ACB = \angle DFE$. This proves our claim. □

PROPOSITION 3.28. *CHORDS AND ARCS I.*

In equal circles, chords of equal length divide the circumferences into arcs, and these arcs are respectively equal; that is, their shorter arcs are equal and their longer arcs are equal.

PROOF. Construct equal circles ($\bigcirc O$, $\bigcirc H$) with equal chords ($\overline{AB}$, $\overline{DE}$). We claim that $\overline{AB}$ and $\overline{DE}$ divide the circumferences of $\bigcirc O$ and $\bigcirc H$, respectively, so that their shorter arcs are equal and their longer arcs are equal.

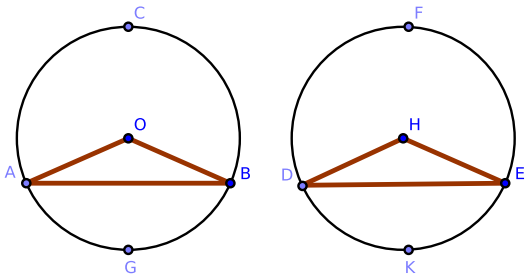


FIGURE 3.2.33. [3.28]

If the equal chords are diameters, the proof follows.

Otherwise, construct $\overline{AO}$, $\overline{OB}$, $\overline{DH}$, and $\overline{HE}$. Because the circles are equal in all respects, their radii are equal [Def. 3.1].

Consider $\triangle AOB$ and $\triangle DHE$: $\overline{AO} = \overline{DH}$, $\overline{OB} = \overline{HE}$, and $\overline{AB} = \overline{DE}$. By [1.8], $\triangle AOB \cong \triangle DHE$; hence $\angle AOB = \angle DHE$, and so arc $AGB =$ arc DKE [3.26].

Since the whole circumference $AGBC$ is equal in length to the whole circumference $DKEF$, it follows that arc $ACB =$ arc DFE. This proves our claim. $\square$

PROPOSITION 3.29. *CHORDS AND ARCS II.*

In equal circles, equal arcs stand opposite equal chords.

PROOF. Construct equal circle $\bigcirc O$ and $\bigcirc H$ where arc AGB =arc DKE. We claim that these arcs stand opposite equal chords.

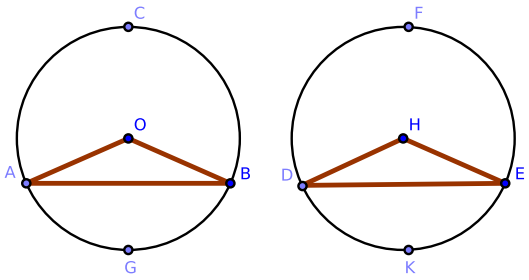


FIGURE 3.2.34. [3.29]

Construct $\overline{AO}$, $\overline{OB}$, $\overline{DH}$, and $\overline{HE}$. Because the circles are equal in all respects, the angles $\angle AOB$ and $\angle DHE$ at the centers which stand on the equal arcs AGB and DKE are themselves equal [3.27].

Consider $\triangle AOB$ and $\triangle DHE$: $\overline{AO} = \overline{DH}$, $\overline{OB} = \overline{HE}$, and $\angle AOB = \angle DHE$. By [1.4], $\triangle AOB \cong \triangle DHE$. Therefore, $\overline{AB} = \overline{DE}$, which proves our claim. □

COROLLARY. *3.29.1. Propositions [3.26]-[3.29] are related in the following sense:*

1. In [3.26], if we have equal angles, then we have equal arcs.

2. In [3.27], if we have equal arcs, then we have equal angles. [3.27] is the converse of [3.26], and together state that we have equal angles if and only if we have equal arcs.

3. In [3.28], if we have equal chords, then we have equal arcs.

4. In [3.29], if we have equal arcs, then we have equal chords. [3.29] is the converse of [3.28], and together state that we have equal chords if and only if we have equal arcs.

Or: we have equal chords if and only if we have equal angles if and only if the angles stand on equal arcs.

REMARK. Since the two circles in the four last propositions are equal, they are congruent figures, and the truth of the propositions is made evident by superposition.

PROPOSITION 3.30. *BISECTING AN ARC.*

PROOF. We wish to bisect the given arc ACB.

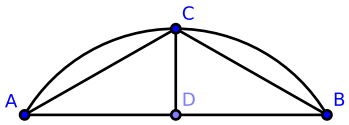


FIGURE 3.2.35. [3.30]

Construct the chord $\overline{AB}$ and bisect it at D. Construct $\overline{DC} \perp \overline{AB}$, intersecting the arc at C. We claim that arc ACB is bisected at C.

Construct $\overline{AC}$ and $\overline{BC}$, and consider $\triangle ADC$ and $\triangle BDC$: $\overline{AD} = \overline{DB}$, they share side $\overline{DC}$, and $\angle ADC = \angle BDC$. By [1.4], $\triangle ADC \cong \triangle BDC$, and so $\overline{AC} = \overline{BC}$.

By [3.28], arc $AC =$ arc BC; since arc $ACB = AC \oplus BC$, arc ACB is bisected at C. □

Exercises.

1. Suppose that $ABCD$ is a semicircle whose diameter is $\overline{AD}$ and that the chord $\overline{BC}$ when extended meets $\overline{AE}$ (where $\overline{AE}$ is the extension of $\overline{AD}$). Prove that if $\overline{CE}$ is equal in length to the radius, the arc $\overline{AB} = 3 \cdot \overline{CD}$. [See the final chapter for a solution.]

2. The internal and the external bisectors of the vertical angle of a triangle inscribed in a circle meet the circumference again at points equidistant from the endpoints of the base.

3. If A is one of the points of intersection of two given circles and two chords $\overline{ACD}$, $\overline{ACD}$ are constructed, cutting the circles in the points C, D, C', and D', then the triangles $\triangle BCD$, $\triangle BC'D'$ formed by joining these to the second point B of intersection of the circles are equiangular.

4. If the vertical angle $\angle ACB$ of a triangle inscribed in a circle is bisected by a line $\overleftrightarrow{CD}$ which meets the circle again at D, and from D perpendiculars $\overline{DE}$, $\overline{DF}$ are constructed to the sides, one of which is extended, prove that $\overline{EA} = \overline{BF}$ and hence that $\overline{CE} = \frac{1}{2}(\overline{AC} + \overline{BC})$.

PROPOSITION 3.31. *THALES' THEOREM.*

In a circle,

(1) if a circle is divided into two semicircles, then the angle contained in either arc is a right angle;

(2) if a circle is divided into two unequal arcs and an angle is contained in the larger of the two arcs, then the angle contained in that arc is an acute angle;

(3) if a circle is divided into two unequal arcs and an angle is contained in the smaller of the two arcs, then the angle contained in that arc is an obtuse angle.

PROOF. Construct $\bigcirc O$, diameter $\overline{AB}$, and points C, D, and E such that C and D are on one semicircle and E is on the other semicircle. We claim that:

(1) $\angle ACB$ in arc ACB is a right angle;

(2) $\angle ACE$ in arc ACE is an acute angle.

(3) $\angle ACD$ in arc ACD is an obtuse angle.

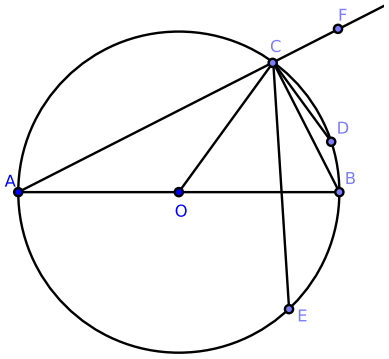


FIGURE 3.2.36. [3.31]

Claim 1: $\angle ACB$ in arc ACB is a right angle;

Construct radius $\overline{OC}$ and extend $\overline{AC}$ to $\overrightarrow{AC}$. Consider $\triangle AOC$: since $\overline{AO} = \overline{OC}$, $\angle ACO = \angle OAC$. Similarly in $\triangle OCB$, $\angle OCB = \angle CBO$. Hence,

$$\begin{aligned} \angle ACB &= \angle ACO + \angle OCB \\ &= \angle OAC + \angle CBO \\ &= \angle BAC + \angle CBA \end{aligned}$$

Consider $\triangle ABC$: by [1.32], $\angle FCB = \angle BAC + \angle CBA$. Hence, $\angle ACB = \angle FCB$ where each are adjacent angles; thus, $\angle ACB$ is a right angle.

Claim 2: $\angle ACE$ in arc ACE is an acute angle.

Construct $\overline{CE}$. Since $\angle ACB = \angle ACE + \angle BCE$, $\angle ACB > \angle ACE$. But we have proven that $\angle ACB$ is a right angle; thus, $\angle ACE$ is acute.

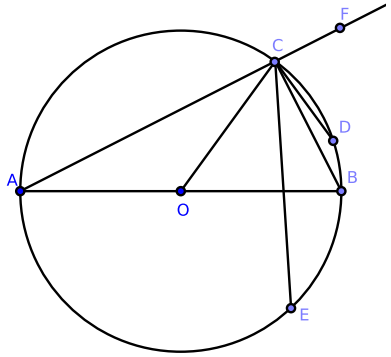


FIGURE 3.2.37. [3.31]

Claim 3: $\angle ACD$ in arc ACD is an obtuse angle;

Construct $\overline{CD}$. Since $\angle ACD = \angle ACB + \angle BCD$, $\angle ACD > \angle ACB$. Since $\angle ACB$ is a right angle, $\angle ACE$ is obtuse. □

COROLLARY. *3.31.1. If a parallelogram is inscribed in a circle, its diagonals intersect at the center of the circle.*

COROLLARY. *3.31.2. [3.31] holds if arcs are replaced by chords of appropriate length, mutatis mutandis.*

PROPOSITION 3.32. *TANGENT-CHORD ANGLES RELATED TO ANGLES ON THE CIRCUMFERENCE WHICH STAND ON THE CHORD.*

If a line is tangent to a circle, and from the point of intersection a chord is constructed cutting the circle, the angles made by this chord with the tangent are respectively equal to the angles in the alternate arcs of the circle.

PROOF. Construct $\overleftrightarrow{EF}$ such that $\overleftrightarrow{EF}$ is tangent to $\bigcirc ABC$ at A. Construct chord $\overline{AC}$; notice that $\overline{AC}$ cuts $\bigcirc ABC$. We claim that the angles made by this chord with the tangent are respectively equal to the angles in the alternate arcs of the circle. We shall prove this in two cases.

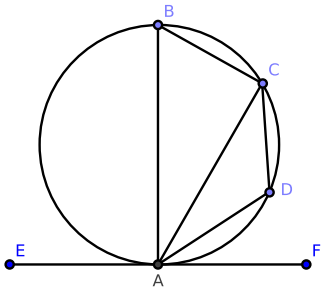


FIGURE 3.2.38. [3.32](α)

Case 1: construct the figures from Figure (α). We wish to show that $\angle ABC = \angle FAC$.

Construct $\overline{AB}$ such that $\overline{AB} \perp \overleftrightarrow{EF}$. Also construct $\overline{BC}$. Because $\overleftrightarrow{EF}$ is tangent to $\bigcirc ABC$, $\overline{AB}$ is clearly constructed at A, and $\overline{AB} \perp \overleftrightarrow{EF}$, by [3.19] we have that $\overline{AB}$ passes through the center of $\bigcirc ABC$. By [3.31], $\angle ACB$ is a right angle; since $\triangle ABC$ is a triangle, the sum of the two remaining angles, $\angle ABC + \angle CAB$, equals one right angle.

Since the angle $\angle BAF$ is right by construction, we have that $\angle ABC + \angle CAB = \angle BAF$. From this we obtain $\angle ABC = \angle BAF - \angle CAB = \angle FAC$, which proves our claim.

Case 2: construct the figures from Figure (β).

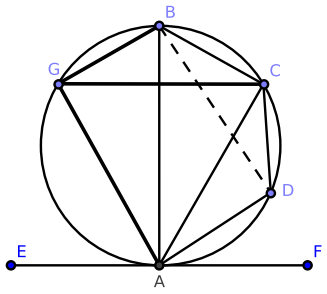


FIGURE 3.2.39. [3.32](β)

Take any point D on the arc AC. We wish to prove that $\angle CAE = \angle CDA$.

Since the quadrilateral $ABCD$ is cyclic, the sum of the opposite angles $\angle ABC + \angle CDA$ equals two right angles [3.22] and is therefore equal to the sum $\angle FAC + \angle CAE$. However, $\angle ABC = \angle FAC$ by case 1. Hence, $\angle CDA = \angle CAE$.

The proves our claim. □

An alternative proof to case 2:

PROOF. Construct the figures from Figure (β). Take any point G in the semicircle AGB. Construct $\overline{AG}$, $\overline{GB}$, $\overline{GC}$.

We then have that $\angle AGB = \angle FAB$, since each angle is right, and $\angle CGB = \angle CAB$ [3.21]. Since

$$\begin{aligned} \angle AGB - \angle CGB &= \angle AGC \\ \angle FAB - \angle CAB &= \angle FAC \end{aligned}$$

we have that $\angle AGC = \angle FAC$.

Again, construct $\overline{BD}$, $\overline{CD}$. Then $\angle BDA = \angle BAE$, since each angle is right, and $\angle CDB = \angle CAB$ [3.21]. Since

$$\begin{aligned} \angle BDA + \angle CDB &= \angle BAE + \angle CAB \\ \angle CDA &= \angle CAE \end{aligned}$$

This proves our claim. □

Exercises.

1. If two circles touch, any line constructed through the point of intersection will cut off similar segments.

2. If two circles touch and any two lines are constructed through the point of intersection (cutting both circles again), the chord connecting their points of intersection with one circle is parallel to the chord connecting their points of intersection with the other circle.

3. Suppose that ACB is an arc of a circle, $\overleftrightarrow{CE}$ a tangent at C (meeting the chord $\overline{AB}$ extended to E), and $\overline{AD} \perp \overline{AB}$ where D is a point of $\overline{AB}$. Prove that if $\overline{DE}$ be bisected at C then the arc $\overline{AC} = 2 \cdot \overline{CB}$.

4. If two circles touch at a point A and if $\overline{ABC}$ is a chord through A, meeting the circles at points B and C, prove that the tangents at B and C are parallel to each other, and that when one circle is within the other, the tangent at B meets the outer circle at two points equidistant from C.

5. If two circles touch externally, their common tangent at either side stands opposite a right angle at the point of intersection, and its square is equal to the rectangle contained by their diameters.

PROPOSITION 3.33. *CONSTRUCTING ARCS GIVEN A SEGMENT AND AN ANGLE.*

It is possible to construct an arc of a circle on a given segment such that the arc contains an angle equal to a given angle.

PROOF. Construct $\bigcirc ABC$, chord $\overline{AB}$, tangent $\overline{AE}$, and $\angle FGH$. We wish to construct $\angle ACB$ such that $\angle ACB = \angle FGH$.

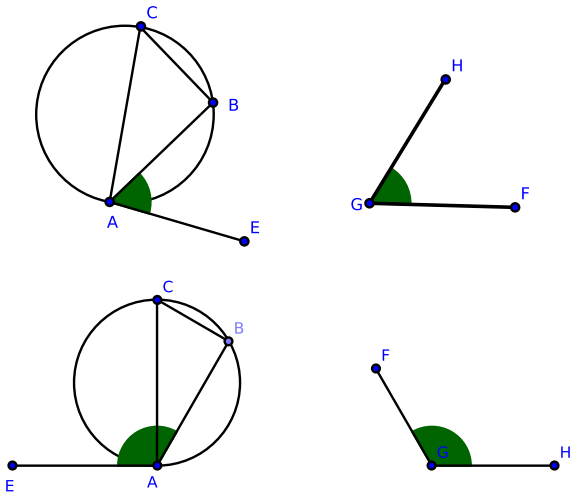


FIGURE 3.2.40. [3.33]

If $\angle FGH$ is a right angle, construct $\overline{AB}$ to be a diameter of $\bigcirc ABC$. The proof follows since the angle in a semicircle is a right angle.

Otherwise construct the angle $\angle BAE$ on $\overline{AB}$ such that $\angle BAE = \angle FGH$. Construct $\overline{AC} \perp \overline{AE}$ and $\overline{BC} \perp \overline{AB}$. We claim that $\angle ACB$ is the required angle.

Since $\angle CAE$ is right, $\bigcirc ABC$ touches $\overline{AE}$ by [3.16]. Therefore, $\angle BAE = \angle ACB$ by [3.32]. But $\angle BAE = \angle FGH$ by construction, and so $\angle FGH = \angle ACB$. This completes our construction. □

Exercises.

1. Construct a triangle, being given the base, vertical angle, and any of the following data:

   (a) a perpendicular.

(b) the sum or difference of the sides.

(c) the sum or difference of the squares of the sides.

(d) the side of the inscribed square on the base.

(e) the median that bisects the base.

2. If lines are constructed from a fixed point to all the points of the circumference of a given circle, prove that the locus of all their points of bisection is a circle.

3. Given the base and vertical angle of a triangle, find the locus of the midpoint of the line joining the vertices of equilateral triangles constructed on the sides.

4. In the same case, find the loci of the vertices of a square constructed on one of the sides.

PROPOSITION 3.34. *DIVIDING ARCS GIVEN AN ANGLE.*

It is possible to divide the circumference of a circle such that the separated arc contains an angle equal to a given angle.

PROOF. From a given circle ($\bigcirc BCA$), we wish to cut off an arc (AC) which contains an angle equal to a given angle ($\angle FGH$).

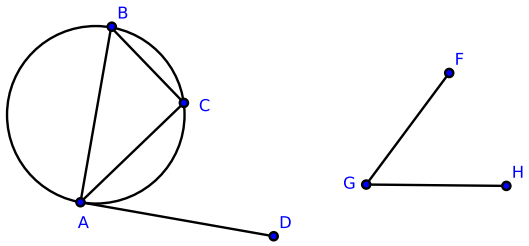


FIGURE 3.2.41. [3.34]

Take any point A on the circumference and construct the tangent $\overline{AD}$. Construct the angle $\angle DAC$ such that $\angle DAC = \angle FGH$. Take any point B on the alternate arc, and construct $\overline{BA}$ and $\overline{BC}$. Then $\angle DAC = \angle ABC$ [3.32]. It follows that $\angle ABC = \angle FGH$, and so the arc AC contains an angle equal to $\angle FGH$. □

PROPOSITION 3.35. *AREAS OF RECTANGLES CONSTRUCTED ON CHORDS.*

If two chords of a circle intersect at one point within the circle, then the area of the rectangle contained by the divided segments of the first chord is equal in area to the rectangle contained by the divided segments of the second chord [Def. 2.4].

PROOF. Construct $\bigcirc O$ with chords $\overline{AB}$ and $\overline{CD}$ which intersect at one point. We claim that the rectangles contained by the segments are equal in area and shall prove this claim in four cases.

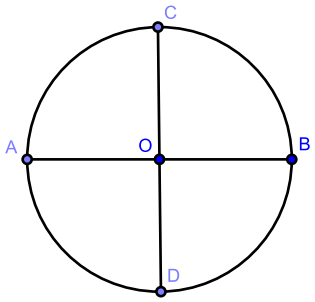


FIGURE 3.2.42. [3.35], case 1

Case 1: if the point of intersection is the center of $\bigcirc O$, each rectangle is equal in area to the square of the radius. The proof follows.

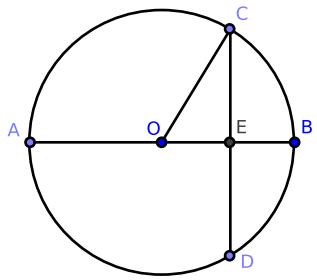


FIGURE 3.2.43. [3.35], case 2

Case 2: suppose that $\overline{AB}$ passes through O and that $\overline{CD}$ does not; also suppose they intersect at E and that $\overline{AB} \perp \overline{CD}$.

Construct $\overline{OC}$. Because $\overline{AB}$ is divided equally at O and unequally at E, by [2.5], we have that

$$\overline{AE} \cdot \overline{EB} + (\overline{OE})^2 = (\overline{OB})^2$$

Since $\overline{OB} = \overline{OC}$,

$$\overline{AE} \cdot \overline{EB} + (\overline{OE})^2 = (\overline{OC})^2$$

But $(\overline{OC})^2 = (\overline{OE})^2 + (\overline{EC})^2$ by [1.47], and therefore

$$\overline{AE} \cdot \overline{EB} + (\overline{OE})^2 = (\overline{OE})^2 + (\overline{EC})^2$$

Subtracting $(\overline{OE})^2$ from both sides of the equality, we obtain

$$\overline{AE} \cdot \overline{EB} = (\overline{EC})^2$$

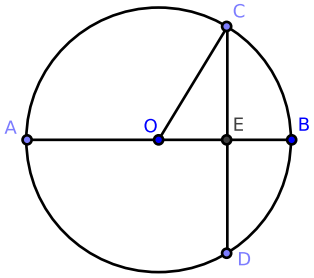


FIGURE 3.2.44. [3.35], case 2

Because $\overline{AB}$ passes through the center and cuts $\overline{CD}$ which does not pass through the center at right angles, $\overline{AB}$ bisects $\overline{CD}$ by [3.3]. So $(\overline{EC})^2 = \overline{CE} \cdot \overline{ED}$, and therefore,

$$\overline{AE} \cdot \overline{EB} = \overline{CE} \cdot \overline{ED}$$

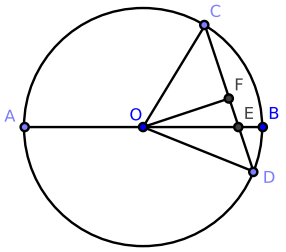


FIGURE 3.2.45. [3.35], case 3

3. Construct diameter $\overline{AB}$ of $\bigcirc O$ which cuts $\overline{CD}$ such that $\overline{AB} \not\perp \overline{CD}$.

Construct $\overline{OC}$, $\overline{OD}$, and $\overline{OF} \perp \overline{CD}$ [1.11]. Since $\overline{CD}$ is cut at right angles by $\overline{OF}$ where $\overline{OF}$ passes through O, $\overline{CD}$ is bisected at F [3.3] and divided unequally at E. By [2.5], $\overline{CE} \cdot \overline{ED} + (\overline{FE})^2 = (\overline{FD})^2$.

Adding $(\overline{OF})^2$ to each side of the equality, we obtain:

$$\begin{aligned} \overline{CE}\cdot\overline{ED}+(\overline{FE})^2+(\overline{OF})^2 &= (\overline{FD})^2+(\overline{OF})^2 \\ \overline{CE}\cdot\overline{ED}+(\overline{OE})^2 &= (\overline{OD})^2 \\ \overline{CE}\cdot\overline{ED}+(\overline{OE})^2 &= (\overline{OB})^2 \end{aligned}$$

Again, since AB is bisected at O and divided unequally at E, $\overline{AE}\cdot\overline{EB}+(\overline{OE})^2 = (\overline{OB})^2$ [2.5].

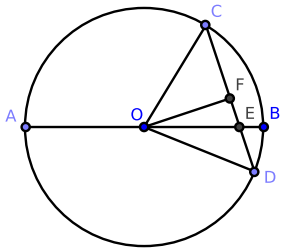


FIGURE 3.2.46. [3.35], case 3

It follows that

$$\begin{aligned} \overline{CE}\cdot\overline{ED}+(\overline{OE})^2 &= \overline{AE}\cdot\overline{EB}+(\overline{OE})^2 \\ \overline{CE}\cdot\overline{ED} &= \overline{AE}\cdot\overline{EB} \end{aligned}$$

4. Suppose neither chord passes through the center and that they intersect at E.

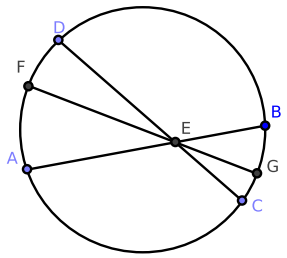


FIGURE 3.2.47. [3.35], case 4

Through E, construct diameter $\overline{FG}$. By case 3, the rectangle $\overline{FE}\cdot\overline{EG} = \overline{AE}\cdot\overline{EB}$ and $\overline{FE}\cdot\overline{EG} = \overline{CE}\cdot\overline{ED}$. Hence, $\overline{AE}\cdot\overline{EB} = \overline{CE}\cdot\overline{ED}$. □

COROLLARY. *3.35.1. If a chord of a circle is divided at any point within the circle, the rectangle contained by its segments is equal to the difference between the square of the radius and the square of the segment constructed from the center to the point of section.*

COROLLARY. *3.35.2. If the rectangle contained by the segments of one of two intersecting segments is equal to the rectangle contained by the segments of the other, the four endpoints are concyclic.*

COROLLARY. *3.35.3. If two triangles are equiangular, the rectangle contained by the non-corresponding sides about any two equal angles are equal.*

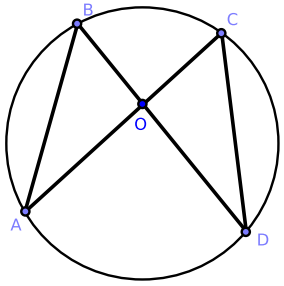


FIGURE 3.2.48. [3.35], Cor. 3

PROOF. Let $\triangle ABO$ and $\triangle DCO$ be the equiangular triangles, and let them be placed so that the equal angles at O are vertically opposite and that the non-corresponding sides, $\overline{AO}$ and $\overline{CO}$, form segment $\overline{CA}$. Then the non-corresponding sides BO, OD form segment $\overline{BD}$. Since $\angle ABD = \angle ACD$, the points A, B, C, D are concyclic [3.21, Cor. 1]. Hence, $\overline{AO} \cdot \overline{OC} = \overline{BO} \cdot \overline{OD}$ [3.35]. □

Exercises.

1. In any triangle, the rectangle contained by two sides is equal in area to the rectangle contained by the perpendicular on the third side and the diameter of the circumscribed circle.

Definition: 1. The supplement of an arc is the amount by which an arc is less than a semicircle, or an angle less than two right angles.

2. The rectangle contained by the chord of an arc and the chord of its supplement is equal to the rectangle contained by the radius and the chord of twice the supplement.

3. If the base of a triangle is given with the sum of the sides, the rectangle contained by the perpendiculars from the endpoints of the base on the external bisector of the vertical angle is given.

4. If the base and the difference of the sides is given, the rectangle contained by the perpendiculars from the endpoints of the base on the internal bisector is given.

5. Through one of the points of intersection of two circles, construct a secant so that the rectangle contained by the intercepted chords may be given, or is a maximum.

6. If we join AF, BF, we find that the rectangle $AF \cdot FB$ is equal in area to the rectangle contained by the radius and $2 \cdot FI$; that is, it is equal to the rectangle contained by the radius and the sum of CF and BG. Hence, if the sum of two arcs of a circle is greater than a semicircle, the rectangle contained by their chords is equal to the rectangle contained by the radius and the sum of the chords of the supplements of their sum and their difference.

7. Through a given point, construct a transversal cutting two given lines so that the rectangle contained by the segments intercepted between it and the line may be given.

8. If the sum of two arcs AC, CB of a circle is less than a semicircle, the rectangle $\overline{AC} \cdot \overline{CB}$ contained by their chords is equal in area to the rectangle contained by the radius and the excess of the chord of the supplement of their difference above the chord of the supplement of their sum.

PROPOSITION 3.36. *THE AREA OF RECTANGLES CONSTRUCTED ON A TANGENT AND A POINT OUTSIDE THE CIRCLE I.*

Suppose we are given a circle and a point outside of the circle. If two segments are constructed from the point to the circle, the first of which intersects the circle at two points and the second of which is tangent to the circle, then the area of the rectangle contained by the subsegments of the first segment is equal to the square on the tangent.

PROOF. Construct $\bigcirc O$ and point P outside of $\bigcirc O$. Then construct $\overline{PT}$ tangent to $\bigcirc O$ at T; also construct $\overline{PA}$ such that $\overline{PA}$ intersects the circle at B and again at A. We claim that $\overline{AP} \cdot \overline{BP} = (\overline{PT})^2$.

We prove this claim in two cases.

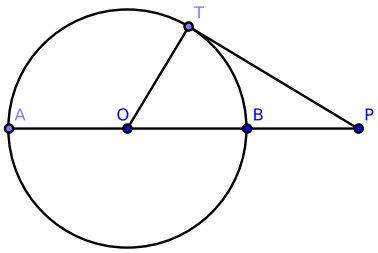


FIGURE 3.2.49. [3.36], case 1

Case 1: $\overline{PA}$ passes through O.

Construct $\overline{OT}$. Because $\overline{AB}$ is bisected at O and divided externally at P, the rectangle $\overline{AP} \cdot \overline{BP} + (\overline{OB})^2 = (\overline{OP})^2$ [2.6].

Since $\overline{PT}$ is a tangent to $\bigcirc O$ and $\overline{OT}$ is constructed from the center to the point of intersection, the angle $\angle OTP$ is right [3.18]. Hence $(\overline{OT})^2 + (\overline{PT})^2 = (\overline{OP})^2$ by [1.47].

Therefore $\overline{AP} \cdot \overline{BP} + (\overline{OB})^2 = (\overline{OT})^2 + (\overline{PT})^2$. But $(\overline{OB})^2 = (\overline{OT})^2$, and so $\overline{AP} \cdot \overline{BP} = (\overline{PT})^2$.

Case 2: $\overline{PA}$ does not pass through O.

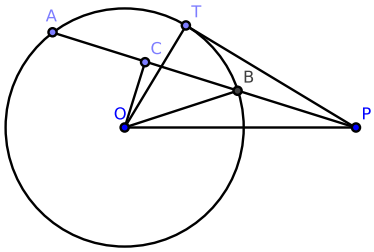


FIGURE 3.2.50. [3.36], case 2

Construct the perpendicular $\overline{OC} \perp \overline{AB}$; also construct $\overline{OT}$, $\overline{OB}$, $\overline{OP}$. Because $\overline{OC}$, a segment through the center, cuts $\overline{AB}$, which does not pass through the center at right angles, $\overline{OC}$ bisects $\overline{AB}$ [3.3].

Since $\overline{AB}$ is bisected at C and divided externally at P, we have that $\overline{AP} \cdot \overline{BP} + (\overline{CB})^2 = (\overline{CP})^2$ [2.6]. Adding $(\overline{OC})^2$ to each side, we obtain:

$$\begin{aligned} \overline{AP} \cdot \overline{BP} + (\overline{CB})^2 + (\overline{OC})^2 &= (\overline{CP})^2 + (\overline{OC})^2 \\ \overline{AP} \cdot \overline{BP} + (\overline{OB})^2 &= (\overline{OP})^2 \end{aligned}$$

We also have that $(\overline{OT})^2 + (\overline{PT})^2 = (\overline{OP})^2$, from which it follows that

$$\overline{AP} \cdot \overline{BP} + (\overline{OB})^2 = (\overline{OT})^2 + (\overline{PT})^2$$

Since $\overline{OB} = \overline{OT}$, $(\overline{OB})^2 = (\overline{OT})^2$, and so $\overline{AP} \cdot \overline{BP} = (\overline{PT})^2$. □

REMARK. The two propositions [3.35] and [3.36] may be written as one statement: the rectangle $\overline{AP} \cdot \overline{BP}$ contained by the segments of any chord of a given circle passing through a fixed point P, either within or outside of the circle, is constant.

PROOF. Suppose O is the center the circle, and construct $\overline{OA}$, $\overline{OB}$, $\overline{OP}$. Notice that $\triangle OAB$ is an isosceles triangle, and $\overline{OP}$ is a segment constructed from its vertex to a point P in the base or the extended base.

It follows that the rectangle $\overline{AP} \cdot \overline{BP}$ is equal to the difference of the squares of $\overline{OB}$ and $\overline{OP}$; therefore, it is constant. □

COROLLARY. *3.36.1. If two segments $\overline{AB}$, $\overline{CD}$ are extended to meet at P, and if the rectangle $\overline{AP} \cdot \overline{BP} = \overline{CP} \cdot \overline{DP}$, then the points A, B, C, D are concyclic (compare [3.35, Cor. 2]).*

COROLLARY. *3.36.2. Tangents to two circles from any point in their common chord are equal (compare [3.17, #6]).*

COROLLARY. *3.36.3. The common chords of any three intersecting circles are concurrent (compare [3.17, #7]).*

Exercises.

1. If the segment $\overline{AD}$ is constructed from the vertex A of $\triangle ABC$ which then intersects $\overline{CB}$ extended to D and creates the angle $\angle BAD = \angle ACB$, prove that $\overline{DB} \cdot \overline{DC} = (\overline{DA})^2$.

2. Prove [Cor. 3.36.1].

PROPOSITION 3.37. *THE AREA OF RECTANGLES CONSTRUCTED ON A TANGENT AND A POINT OUTSIDE THE CIRCLE II.*

Suppose we are given a circle and a point outside of the circle. If two segments are constructed from the point to the circle, the first of which intersects the circle at two points, and the area of the rectangle contained by the subsegments of the first segment is equal to the square on the second segment, then the second segment is tangent to the circle.

PROOF. If the rectangle ($\overline{AP} \cdot \overline{BP}$) contained by the segments of a secant and constructed from any point (P) outside of the circle ($\bigcirc O$) is equal in area to the square on the segment ($\overline{PT}$) constructed from the same point to meet the circle, we claim that the segment which meets the circle is a tangent to that circle.

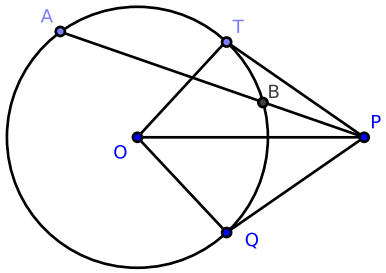


FIGURE 3.2.51. [3.37]

From P, construct $\overline{PQ}$ touching $\bigcirc O$ [3.17]. Construct $\overline{OP}$, $\overline{OQ}$, $\overline{OT}$. By hypothesis, $\overline{AP} \cdot \overline{BP} = (\overline{PT})^2$; by [3.36], $\overline{AP} \cdot \overline{BP} = (\overline{PQ})^2$. Hence $(\overline{PT})^2 = (\overline{PQ})^2$, and so $\overline{PT} = \overline{PQ}$.

Consider the triangles $\triangle OTP$ and $\triangle OQP$: each have $\overline{OT} = \overline{OQ}$, $\overline{TP} = \overline{QP}$, and they share base $\overline{OP}$ in common. By [1.8], $\triangle OTP \cong \triangle OQP$, and so $\angle OTP = \angle OQP$.

But $\angle OQP$ is a right angle since $\overline{PQ}$ is a tangent [3.38]; hence $\angle OTP$ is right, and therefore $\overline{PT}$ is a tangent to $\bigcirc O$ [3.16]. □

COROLLARY. *3.37.1. Suppose we are given a circle and a point outside of the circle where two segments are constructed from the point to the circle, the first of which intersects the circle at two points. Then the second segment is tangent to the circle if and only if the area of the rectangle contained by the subsegments of the first segment is equal to the square on the tangent.*

Exercises.

1. Construct a circle passing through two given points and fulfilling either of the following conditions:

   (a) touching a given line;

   (b) touching a given circle.

2. Construct a circle through a given point and touching two given lines; or touching a given line and a given circle.

3. Construct a circle passing through a given point having its center on a given line and touching a given circle.

4. Construct a circle through two given points and intercepting a given arc on a given circle.

5. If A, B, C, D are four collinear points and $\overline{EF}$ is a common tangent to the circles constructed upon $\overline{AB}$, $\overline{CD}$ as diameters, then prove that the triangles $\triangle AEB$, $\triangle CFD$ are equiangular.

6. The diameter of the circle inscribed in a right-angled triangle is equal to half the sum of the diameters of the circles touching the hypotenuse, the perpendicular from the right angle of the hypotenuse, and the circle constructed about the right-angled triangle.

Exam questions for chapter 3.

1. What is the subject-matter of chapter 3?

2. Define equal circles.

3. Define a chord.

4. When does a secant become a tangent?

5. What is the difference between an arc and a sector?

6. What is meant by an angle in a segment?

7. If an arc of a circle is one-sixth of the whole circumference, what is the magnitude of the angle in it?

8. What are segments?

9. What is meant by an angle standing on a segment?

10. What are concyclic points?

11. What is a cyclic quadrilateral?

12. How many intersections can a line and a circle have?

13. How many points of intersection can two circles have?

14. Why is it that if two circles touch they cannot have any other common point?

15. State a proposition that encompasses [3.11] and [3.12].

16. What proposition is #16 a limiting case of?

17. What is the modern definition of an angle?

18. How does the modern definition of an angle differ from Euclid's?

19. State the relations between [3.16], [3.18] and [3.19].

20. What propositions are [3.16], [3.18] and [3.19] limiting cases of?

21. How many common tangents can two circles have?

22. What is the magnitude of the rectangle of the segments of a chord constructed through a point 3.65m distant from the center of a circle whose radius is 4.25m?

23. The radii of two circles are 4.25 and 1.75 ft respectively, and the distance between their centers 6.5 ft. Find the lengths of their direct and their transverse common tangents.

24. If a point is h feet outside the circumference of a circle whose diameter is 7920 miles, prove that the length of the tangent constructed from it to the circle is $\sqrt{3h/2}$ miles.

25. Two parallel chords of a circle are 12 inches and 16 inches respectively and the distance between them is 2 inches. Find the length of the diameter.

26. What is the locus of the centers of all circles touching a given circle in a given point?

27. What is the condition that must be fulfilled that four points may be concyclic?

28. If the angle in a segment of a circle equals 1.5 right angles, what part of the whole circumference is it?

29. Mention the converse propositions of chapter 3 which are proved directly.

30. What is the locus of the midpoints of equal chords in a circle?

31. The radii of two circles are 6 and 8, and the distance between their centers is 10. Find the length of their common chord.

32. If a figure of any even number of sides is inscribed in a circle, prove that the sum of one set of alternate angles is equal to the sum of the remaining angles.

Chapter 3 exercises.

1. If two chords of a circle intersect at right angles, the sum of the squares on their segments is equal to the square on the diameter.

2. If a chord of a given circle stands opposite a right angle at a fixed point, the rectangle of the perpendiculars on it from the fixed point and from the center of the given circle is constant. Also, the sum of the squares of perpendiculars on it from two other fixed points (which may be found) is constant.

3. If through either of the points of intersection of two equal circles any line is constructed meeting them again in two points, these points are equally distant from the other intersection of the circles.

4. Construct a tangent to a given circle so that the triangle formed by it and two fixed tangents to the circle shall be:

(a) a maximum;

(b) a minimum.

5. If through the points of intersection A, B of two circles any two segments $\overline{ACD}$, $\overline{BEF}$ are constructed parallel to each other which meet the circles again at C, D, E, F, then we have that $\overline{CD} = \overline{EF}$.

6. In every triangle, the bisector of the greatest angle is the least of the three bisectors of the angles.

7. The circles whose diameters are the four sides of any cyclic quadrilateral intersect again in four concyclic points.

8. The four vertices of a cyclic quadrilateral determine four triangles whose orthocenters (the intersections of their perpendiculars) form an equal quadrilateral.

9. If through one of the points of intersection of two circles we construct two common chords, the segments joining the endpoints of these chords make a given angle with each other.

10. The square on the perpendicular from any point in the circumference of a circle on the chord of contact of two tangents is equal to the rectangle of the perpendiculars from the same point on the tangents.

11. Find a point on the circumference of a given circle such that the sum of the squares on whose distances from two given points is either a maximum or a minimum.

12. Four circles are constructed on the sides of a quadrilateral as diameters. Prove that the common chord of any two on adjacent sides is parallel to the common chord of the remaining two.

13. The rectangle contained by the perpendiculars from any point in a circle on the diagonals of an inscribed quadrilateral is equal to the rectangle contained by the perpendiculars from the same point on either pair of opposite sides.

14. The rectangle contained by the sides of a triangle is greater than the square on the internal bisector of the vertical angle by the rectangle contained by the segments of the base.

15. If through A, one of the points of intersection of two circles, we construct any line $\overleftrightarrow{ABC}$ which cuts the circles again at B and C, the tangents at B and C intersect at a given angle.

16. If a chord of a given circle passes through a given point, the locus of the intersection of tangents at its endpoints is a straight line.

17. The rectangle contained by the distances of the point where the internal bisector of the vertical angle meets the base and the point where the perpendicular from the vertex meets it from the midpoint of the base is equal to the square on half the difference of the sides.

18. State and prove the proposition analogous to [3.17] for the external bisector of the vertical angle.

19. The square on the external diagonal of a cyclic quadrilateral is equal to the sum of the squares on the tangents from its endpoints to the circumscribed circle.

20. If a "movable" circle touches a given circle and a given line, the chord of contact passes through a given point.

21. If A, B, C are three points in the circumference of a circle, and D, E are the midpoints of the arcs AB, AC, and if the segment $\overline{DE}$ intersects the chords $\overline{AB}$, $\overline{AC}$ at F and G, then $\overline{AF} = \overline{AG}$.

22. If a cyclic quadrilateral is such that a circle can be inscribed in it, the lines joining the points of contact are perpendicular to each other.

23. If through the point of intersection of the diagonals of a cyclic quadrilateral the minimum chord is constructed, that point will bisect the part of the chord between the opposite sides of the quadrilateral.

24. Given the base of a triangle, the vertical angle, and either the internal or the external bisector at the vertical angle, construct the triangle.

25. If through the midpoint A of a given arc BAC we construct any chord $\overline{AD}$, cutting BC at E, then the rectangle $\overline{AD} \cdot \overline{AE}$ is constant.

26. The four circles circumscribing the four triangles formed by any four lines pass through a common point.

27. If X, Y, Z are any three points on the three sides of a triangle $\triangle ABC$, the three circles about the triangles $\triangle YAZ$, $\triangle ZBX$, $\triangle XCY$ pass through a common point.

28. If the position of the common point in the previous exercise are given, the three angles of the triangle $\triangle XYZ$ are given, and conversely.

29. Place a given triangle so that its three sides shall pass through three given points.

30. Place a given triangle so that its three vertices shall lie on three given lines.

31. Construct the largest triangle equiangular to a given one whose sides shall pass through three given points.

32. Construct the smallest possible triangle equiangular to a given one whose vertices shall lie on three given lines.

33. Construct the largest possible triangle equiangular to a given triangle whose sides shall touch three given circles.

34. If two sides of a given triangle pass through fixed points, the third touches a fixed circle.

35. If two sides of a given triangle touch fixed circles, the third touches a fixed circle.

36. Construct an equilateral triangle having its vertex at a given point and the endpoints of its base on a given circle.

37. Construct an equilateral triangle having its vertex at a given point and the endpoints of its base on two given circles.

38. Place a given triangle so that its three sides touch three given circles.

39. Circumscribe a square about a given quadrilateral.

40. Inscribe a square in a given quadrilateral.

41. Construct the following circles:

(a) orthogonal (cutting at right angles) to a given circle and passing through two given points;

(b) orthogonal to two others, and passing through a given point;

(c) orthogonal to three others.

42. If from the endpoints of a diameter $\overline{AB}$ of a semicircle two chords $\overline{AD}$, $\overline{BE}$ are constructed which meet at C, we have that $\overline{AC} \cdot \overline{AD} + \overline{BC} \cdot \overline{BE} = (\overline{AB})^2$.

43. If $ABCD$ is a cyclic quadrilateral, and if we construct any circle passing through the points A and B, another through B and C, a third through C and D, and a fourth through D and A, then these circles intersect successively at four other points E, F, G, H, forming another cyclic quadrilateral.

44. If $\triangle ABC$ is an equilateral triangle, what is the locus of the point M, if $\overline{MA} = \overline{MB} + \overline{MC}$?

45. In a triangle, given the sum or the difference of two sides and the angle formed by these sides both in magnitude and position, the locus of the center of the circumscribed circle is a straight line.

46. Construct a circle:

(a) through two given points which bisect the circumference of a given circle;

(b) through one given point which bisects the circumference of two given circles.

47. Find the locus of the center of a circle which bisects the circumferences of two given circles.

48. Construct a circle which bisects the circumferences of three given circles.

49. If $\overline{CD}$ is a perpendicular from any point C in a semicircle on the diameter $\overline{AB}$, $\bigcirc EFG$ is a circle touching $\overline{DB}$ at E, $\overline{CD}$ at F, and the semicircle at G, then prove that:

(a) the points A, F, G are collinear;

(b) $\overline{AC} = \overline{AE}$.

50. Being given an obtuse-angled triangle, construct from the obtuse angle to the opposite side a segment whose square is equal to the rectangle contained by the segments into which it divides the opposite side.

51. If O is a point outside a circle whose center is E and two perpendicular segments passing through O intercept chords $\overline{AB}$, $\overline{CD}$ on the circle, then prove that $(\overline{AB})^2 + (\overline{CD})^2 + 4 \cdot (\overline{OE})^2 = 8 \cdot (\overline{R})^2$.

52. The sum of the squares on the sides of a triangle is equal to twice the sum of the rectangles contained by each perpendicular and the portion of it comprised between the corresponding vertex and the orthocenter. It is also equal to $12 \cdot (\overline{R})^2$ minus the sum of the squares of the distances of the orthocenter from the vertices.

53. If two circles touch at C, if D is any point outside the circles at which their radii through C stands opposite equal angles, and if DE, DF are tangent from D, prove that $\overline{DE} \cdot \overline{DF} = (\overline{CD})^2$.

CHAPTER 4

Inscription and Circumscription

This chapter contains sixteen propositions, four of which relate to triangles, four to squares, four to pentagons, and four to miscellaneous figures.

4.1. Definitions

1. If two polygons are related such that the vertices of one lie on the sides of the other, then:

   (a) the former is said to be *inscribed* in the latter;

   (b) the latter is said to be *circumscribed* around or about the former.

2. A polygon is said to be *inscribed* in a circle when its vertices lie on the circumference. Reciprocally, a polygon is said to be *circumscribed* about or around a circle when each side touches the circle.

3. A circle is said to be *inscribed* in a polygon when it touches each side of the figure. Reciprocally, a circle is said to be *circumscribed* about or around a polygon when it passes through each vertex of the figure.

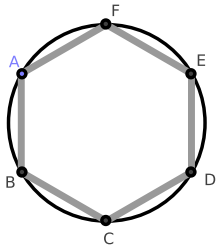


FIGURE 4.1.1. The hexagon is inscribed in the circle, and the circle is circumscribed about the hexagon.

4. A polygon which is both equilateral and equiangular is said to be *regular*.

4.2. Propositions from Book IV

PROPOSITION 4.1. *CONSTRUCTING A CHORD INSIDE A CIRCLE.*

In a given circle, it is possible to construct a chord equal in length to a given segment which is less than the circle's diameter.

PROOF. Construct $\bigcirc ABC$ with diameter $\overline{AC}$; also construct segment $\overline{DG} \leq \overline{AC}$. We wish to construct a chord in $\bigcirc ABC$ equal in length to $\overline{DG}$.

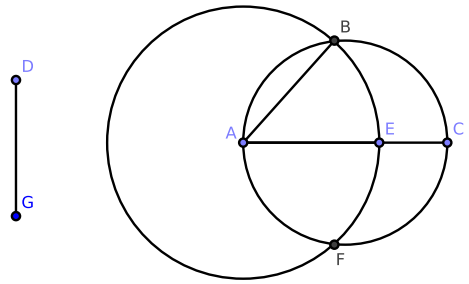


FIGURE 4.2.1. [4.1]

If $\overline{DG} = \overline{AC}$, then the required chord already exists within the circle (i.e., its diameter).

If $\overline{DG} < \overline{AC}$, cut sub-segment $\overline{AE}$ from diameter $\overline{AC}$ such that $\overline{AE} = \overline{DG}$ [1.3]. With A as center and $\overline{AE}$ as radius, construct the circle $\bigcirc A$, cutting the circle $\bigcirc ABC$ at the points B and F. Construct $\overline{AB}$: we claim that $\overline{AB}$ is the required chord.

Notice that $\overline{AB} = \overline{AE}$. But $\overline{AE} = \overline{DG}$ by construction, and so $\overline{AB} = \overline{DG}$. Since $\overline{AB}$ is a chord of $\bigcirc ABC$, our claim is proven. □

PROPOSITION 4.2. *INSCRIBING A TRIANGLE INSIDE A CIRCLE.*

In a given circle, it is possible to inscribe a triangle equiangular to a given triangle.

PROOF. We wish to inscribe a triangle equiangular to $\triangle DEF$ in $\bigcirc ABC$.

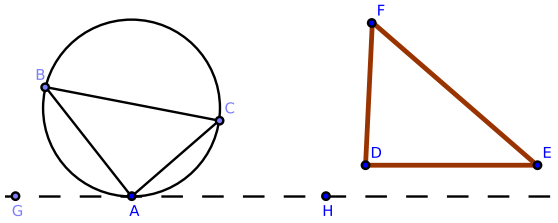


FIGURE 4.2.2. [4.2]

At a point A on the circumference of $\bigcirc ABC$, construct the tangent line $\overleftrightarrow{GAH}$. Construct $\angle HAC = \angle DEF$, $\angle GAB = \angle DFE$ [1.23], and $\overline{BC}$. We claim that $\triangle ABC$ fulfills the required conditions.

Since $\angle DEF = \angle HAC$ by construction and $\angle HAC = \angle ABC$ in the alternate segment [3.32], we have that $\angle DEF = \angle ABC$. Similarly, $\angle DFE = \angle ACB$. By [1.32], $\angle FDE = \angle BAC$.

This proves our claim. □

PROPOSITION 4.3. *CIRCUMSCRIBING A TRIANGLE ABOUT A CIRCLE.*

It is possible to circumscribe a triangle about a circle where the triangle is equiangular to a given triangle.

PROOF. We wish to to construct a triangle equiangular to $\triangle DEF$ about $\bigcirc O$.

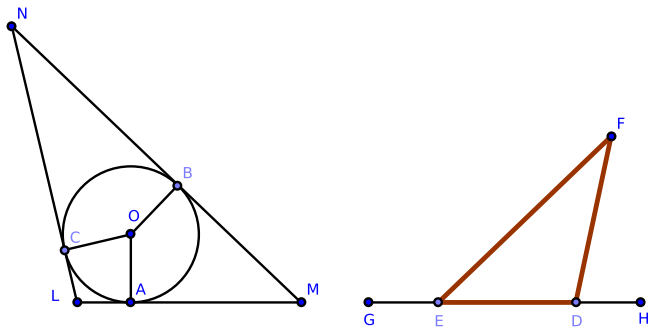


FIGURE 4.2.3. [4.3]

Extend side $\overline{DE}$ of $\triangle DEF$ to $\overline{GH}$, and from O in $\bigcirc O$ construct radius $\overline{OA}$. Construct $\angle AOB = \angle GEF$ and $\angle AOC = \angle HDF$ [1.23]. At the points A, B, and C, construct the tangents $\overline{LM}$, $\overline{MN}$, and $\overline{NL}$ to $\bigcirc O$. We claim that $\triangle LMN$ fulfills the required conditions.

Because $\overline{AM}$ touches $\bigcirc O$ at A, $\angle OAM$ is right [3.18]. Similarly, $\angle MBO$ is right; but the sum of the four angles of the quadrilateral $OAMB$ is equal to four right angles [1.32, Cor. 3]. Therefore the sum of the two remaining angles $\angle AOB + \angle AMB$ equals two right angles.

By [1.13], $\angle GEF + \angle FED =$ two right angles [1.13], and so $\angle AOB + \angle AMB = \angle GEF + \angle FED$. But $\angle AOB = \angle GEF$ by construction; hence $\angle AMB = \angle FED$. Similarly, $\angle ALC = \angle EDF$. By [1.32], $\angle BNC = \angle DFE$, and so $\triangle LMN$ is equiangular to $\triangle DEF$. This proves our claim. □

PROPOSITION 4.4. *INSCRIBING A CIRCLE IN A TRIANGLE.*

It is possible to inscribe a circle in a given triangle.

PROOF. We wish to inscribe $\bigcirc O$ in $\triangle ABC$.

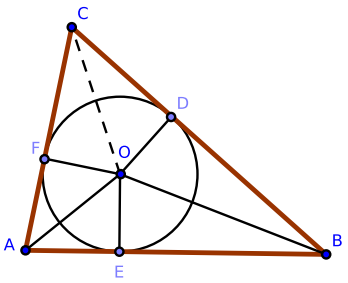


FIGURE 4.2.4. [4.4]

Bisect angles $\angle CAB$ and $\angle ABC$ of $\triangle ABC$ by the segments $\overline{AO}$, $\overline{BO}$; we claim that O, their point of intersection, is the center of the required circle.

From O construct $\overline{OD} \perp \overline{CB}$, $\overline{OE} \perp \overline{AB}$, and $\overline{OF} \perp \overline{AC}$. Consider the triangles $\triangle OAE$ and $\triangle OAF$: $\angle OAE = \angle OAF$ by construction; $\angle AEO = \angle AFO$ since each is right; each triangle shares side $\overline{OA}$. By [1.26], $\triangle OAE \cong \triangle OAF$, and so $\overline{OE} = \overline{OF}$.

Similarly, $\overline{OD} = \overline{OF}$, and so $\overline{OD} = \overline{OE} = \overline{OF}$, and by [3.9], the circle constructed with O as center and $\overline{OD}$ as radius will intersect the points D, E, F.

Since each of the angles $\angle ODB$, $\angle OEA$, $\angle OFA$ is right, each segment touches the respective sides of the triangle $\triangle ABC$ [3.16]. Therefore, the circle $\bigcirc O$ is inscribed in the triangle $\triangle ABC$. □

Definition:

1. The bisectors of the three internal angles of a triangle are concurrent. Their point of intersection is called the *incenter* of the triangle.

Exercises.

1. In [4.4]: if $\overline{OC}$ is constructed, prove that $\angle ACB$ is bisected. Hence, the existence of the *incenter* of a triangle is proven. [See the final chapter for a solution.]

2. If the sides BC, CA, AB of the triangle $\triangle ABC$ are written as a, b, c, and half their sum is written as s, the distances of the vertices A, B, C of the triangle from the points of contact of the inscribed circle are respectively $s-a$, $s-b$, $s-c$.

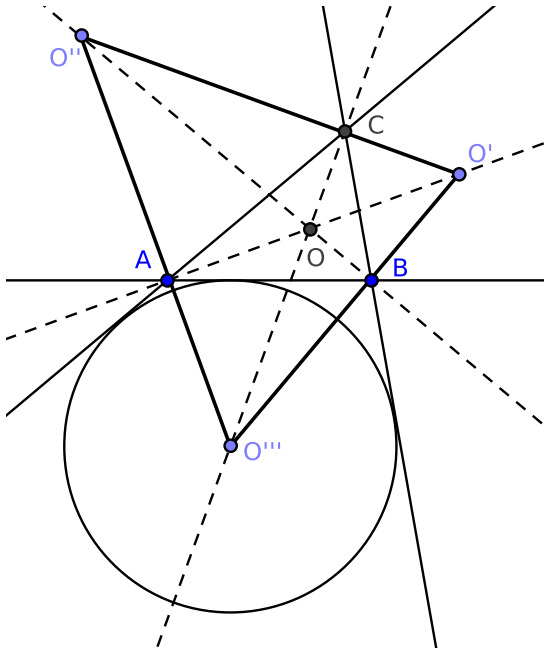


FIGURE 4.2.5. [4.4, #3]

3. If the external angles of the triangle $\triangle ABC$ are bisected as in the above Figure, the three vertices O', O'', O''' of the triangle formed by the three bisectors are the centers of three circles, each touching one side externally and the other two when extended. These three circles are called the *escribed circles* of the triangle $\triangle ABC$.

4. The distances of the vertices A, B, C from the points of contact of the escribed circle which touches AB externally are $s-b$, $s-a$, s.

5. The center of the inscribed circle, the center of each escribed circle, and two of the vertices of the triangle are concyclic. Also, any two of the escribed centers are concyclic with the corresponding two of the vertices of the triangle.

6. Of the four points O, O', O'', O''', any one is the orthocenter of the triangle formed by the remaining three.

7. In [4.4], prove that $\triangle BCO$, $\triangle CAO$, and $\triangle ABO$ are equiangular.

8. Confirm that $CO \cdot CO = ab$, $AO \cdot AO = bc$, $BO \cdot BO = ca$.

9. Since the whole triangle $\triangle ABC$ is made up of the three triangles $\triangle AOB$, $\triangle BOC$, $\triangle COA$, we see that the rectangle contained by the sum of the three sides and the radius of the inscribed circle is equal to twice the area of the triangle. Hence, if r denotes the radius of the inscribed circle, $r \cdot s$ =area of $\triangle ABC$.

10. If r' denotes the radius of the escribed circle which touches the side a externally, it may be shown that $r'(s-a)$ =area of the triangle.

11.Show that $rr' = s - b \cdot s - c$.

12. Show that the square of the area $= s \cdot (s-a) \cdot (s-b) \cdot (s-c)$.

13. Show that the square of the area = $r \cdot r' \cdot r'' \cdot r'''$.

14. If $\triangle ABC$ is a right triangle where the angle at C is right, then $r = s - c$, $r' = s - b$, $r'' = s - a$, and $r''' = s$.

15. Given the base of a triangle, the vertical angle, and the radius of the inscribed or any of the escribed circles, construct it.

PROPOSITION 4.5. *CIRCUMSCRIBING A CIRCLE ABOUT A TRIANGLE.*

It is possible to circumscribe a circle about a given triangle.

PROOF. We wish to construct $\bigcirc O$ about $\triangle ABC$.

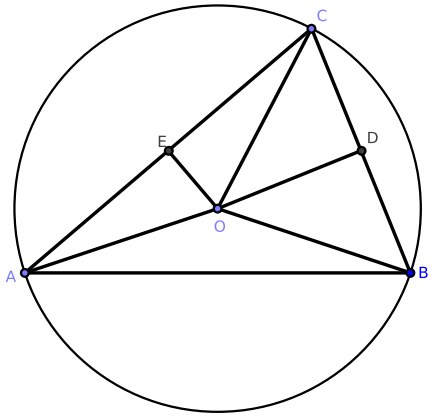


FIGURE 4.2.6. [4.5]

Bisect sides AC and BC of $\triangle ABC$ at the points E and D, respectively. Construct $\overline{DO} \perp \overline{BC}$ and $\overline{EO} \perp \overline{CA}$. We claim that O, the point of intersection of the perpendiculars, is the center of the required circle.

Construct $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and consider triangles $\triangle BDO$ and $\triangle CDO$: sides $BD = CD$ by construction, side DO in common, and $\angle BDO = \angle CDO$ because each is right. By [1.4], $\triangle BDO \cong \triangle CDO$, and so $\overline{BO} = \overline{OC}$.

Similarly, $\overline{AO} = \overline{OC}$. Therefore, $\overline{AO} = \overline{BO} = \overline{CO}$, and by [3.9], the circle is constructed with O as its center and $\overline{OA}$ as its radius, and $\bigcirc O$ will pass through A, B, and C. Thus, $\bigcirc O$ is circumscribed about the triangle $\triangle ABC$. $\square$

COROLLARY. *4.5.1. Since the perpendicular from O to $\overline{AB}$ bisects $\overline{AB}$ by [3.3], we see that the perpendiculars at the midpoints of the sides of a triangle are concurrent. (See the following Definition.)*

Definition:

2. The circle in [4.5] is called the *circumcircle*, its radius the *circumradius*, and its center the *circumcenter* of the triangle.

Exercises.

1. Prove that the three altitudes of a triangle ($\triangle ABC$) are concurrent. (This proves the existence of the *orthocenter* of a circle.) [See the final chapter for a solution.]

2. Prove that the three rectangles $\overline{OA} \cdot \overline{OP}$, $\overline{OB} \cdot \overline{OQ}$, $\overline{OC} \cdot \overline{OR}$ are equal in area.

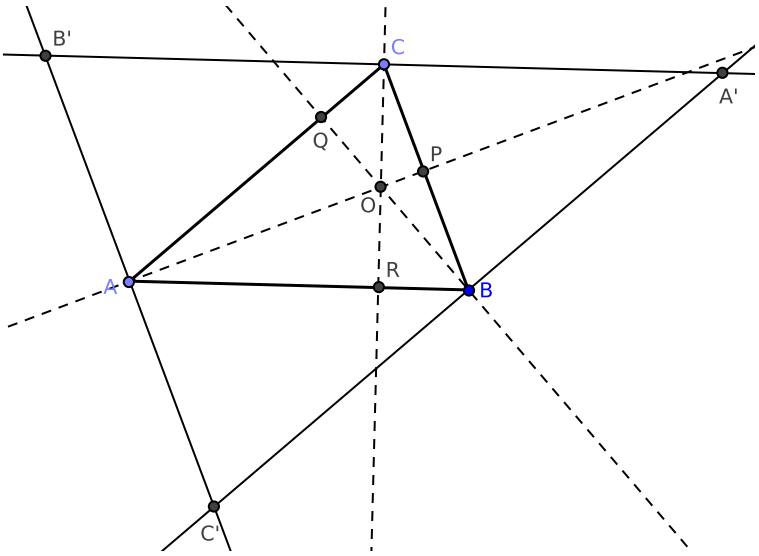


FIGURE 4.2.7. [4.5, #2]

Definition:

3. The circle with O as center, the square of whose radius is equal $\overline{OA} \cdot \overline{OP} = \overline{OB} \cdot \overline{OQ} = \overline{OC} \cdot \overline{OR}$, is called the *polar circle* of the triangle $\triangle ABC$.

3. If the altitudes of a triangle are extended to meet a circumscribed circle, the intercepts between the orthocenter and the circle are bisected by the sides of the triangle.

Definition:

4. The nine-points circle is a circle that can be constructed for any given triangle. It is so named because it passes through nine significant concyclic points defined from the triangle. These nine points are:

(a) the midpoint of each side of the triangle

(b) the foot of each altitude

(c) the midpoint of the line segment from each vertex of the triangle to the orthocenter (where the three altitudes meet; these line segments lie on their respective altitudes). [1] See Fig. 4.2.8.

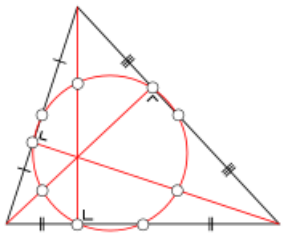


FIGURE 4.2.8. [4.5, #4] The nine-points circle

4. The circumcircle of a triangle is the "nine points circle" of each of the four triangles formed by joining the centers of the inscribed and escribed circles.

5. The distances between the vertices of a triangle and its orthocenter are respectively the doubles of the perpendiculars from the circumcenter on the sides.

6. The radius of the "nine points circle" of a triangle is equal to half its circumradius.

REMARK. The orthocenter, centroid, and circumcenter of any triangle are collinear; the line they lie on is named the Euler line[2].

[1] https://en.wikipedia.org/wiki/Nine-point_circle
[2] https://en.wikipedia.org/wiki/Leonhard_Euler

PROPOSITION 4.6. *INSCRIBING A SQUARE IN A CIRCLE.*

It is possible to to inscribe a square in a given circle.

PROOF. We wish to inscribe the square $\boxdot ABCD$ in $\bigcirc O$.

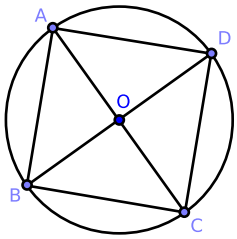


FIGURE 4.2.9. [4.6]

Construct diameters $\overline{AC}$ and $\overline{BD}$ such that $\overline{AC} \perp \overline{BD}$. Also construct $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. We claim that $\boxdot ABCD$ is the required square.

Notice that the four angles at O are equal since they are right angles. Hence the arcs on which they stand are equal [3.26] and the four chords on which they stand are equal in length [3.29]. Therefore the figure $\boxdot ABCD$ is equilateral.

Again, because $\overline{AC}$ is a diameter, the angle $\angle ABC$ is right [3.31]. Similarly, the remaining angles are right. It follows that $\boxdot ABCD$ is a square inscribed in $\bigcirc O$, which proves our claim. □

PROPOSITION 4.7. *CIRCUMSCRIBING A SQUARE ABOUT A CIRCLE.*

It is possible to circumscribe a square about a given circle.

PROOF. We wish to construct the square $\boxdot EHGF$ about $\bigcirc O$.

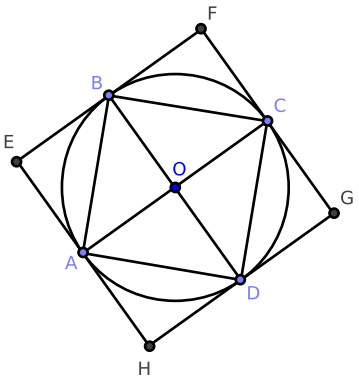


FIGURE 4.2.10. [4.7]

Through O construct diameters $\overline{AC}$ and $\overline{BD}$ such that $\overline{AC} \perp \overline{BD}$, and at the points A, B, C, and D construct the tangential segments $\overline{HE}$, $\overline{EF}$, $\overline{FG}$, and $\overline{GH}$. We claim that $\boxdot EFGH$ is the required square.

Since $\overline{AE}$ touches the circle at A, the angle $\angle EAO$ is right [3.18] and therefore equal to $\angle BOC$, which is right by construction. Thus $\overline{AE} \parallel \overline{OB}$ and $\overline{EB} \parallel \overline{AO}$.

Since $\overline{AO} = \overline{OB}$, the figure $AOBE$ is a rhombus. Since the angle $\angle AOB$ is right, $\boxdot AOBE$ is a square.

Similarly, each of the figures $\boxdot BOFC$, $\boxdot ODGC$, and $\boxdot AHDO$ is a square, and so $\boxdot EHGF$ is a square circumscribed about $\bigcirc O$. This proves our claim. □

COROLLARY. *4.7.1. The circumscribed square, $\boxdot EFGH$, has double the area of the inscribed square, $\boxdot BCDA$.*

Exercises.

1. Prove [Cor. 4.7.1]. [See the final chapter for a solution.]

PROPOSITION 4.8. *INSCRIBING A CIRCLE IN A SQUARE.*

It is possible to inscribe a circle in a given square.

PROOF. We wish to inscribe $\bigcirc O$ in $\boxdot EHGF$.

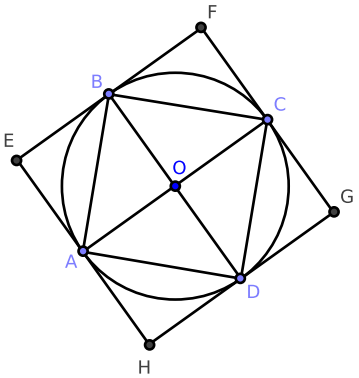


FIGURE 4.2.11. [4.8]

Bisect adjacent sides $\overline{EH}$ and $\overline{EF}$ at A and B; through A and B construct $\overline{AC} \perp \overline{EH}$ and $\overline{BD} \perp \overline{EF}$. We claim that O, the point of intersection of these parallels, is the center of the required circle, $\bigcirc O$.

Because $\boxdot EAOB$ is a parallelogram, its opposite sides are equal; therefore $\overline{OA} = \overline{EB}$. But $\overline{EB}$ is half the side of $\boxdot EHGF$, and so $\overline{OA} =$ half of the side of $\boxdot EHGF$. This is also true for each of the segments $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$, *mutatis mutandis*. Therefore,

$$\overline{OA} = \overline{OB} = \overline{OC} = \overline{OD}$$

By [3.9], O is the center of $\bigcirc O$. And since these segments are perpendicular to the sides of the given square, the circle constructed with O as center and $\overline{OA}$ as radius is inscribed in the square. This proves our claim. □

PROPOSITION 4.9. *CIRCUMSCRIBING A CIRCLE ABOUT A GIVEN SQUARE.*

It is possible to circumscribe a circle about a given square.

PROOF. We wish to construct $\bigcirc O$ about $\boxdot ABCD$.

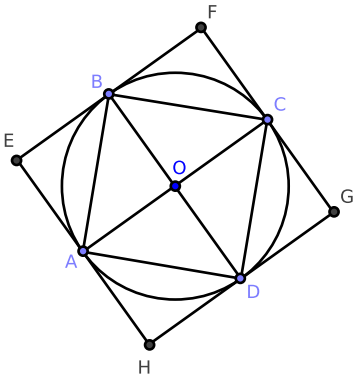


FIGURE 4.2.12. [4.9]

Construct perpendicular diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at O. We claim that O is the center of the required circle.

Since $\triangle ABC$ is an isosceles triangle and the angle $\angle ABC$ is right, each of the other angles equals half a right angle; therefore $\angle BAO = \angle ABO$. By [1.6], $\overline{AO} = \overline{OB}$. Using similar reasoning, we obtain

$$\overline{OA} = \overline{OB} = \overline{OC} = \overline{OD}$$

By [3.9], O is the center of $\bigcirc O$ with radius $= \overline{OA}$ which intersects B, C, and D and is clearly constructed about the square $\boxdot ABCD$. This proves our claim. $\square$

PROPOSITION 4.10. *CONSTRUCTION OF AN ISOSCELES TRIANGLE WITH BASE ANGLES DOUBLE THE VERTICAL ANGLE.*

It is possible to construct an isosceles triangle such that each base angle is double the vertical angle.

PROOF. We wish to construct an isosceles triangle $\triangle ABD$ where

$$\angle BDA = \angle DBA = 2 \cdot \angle DAB$$

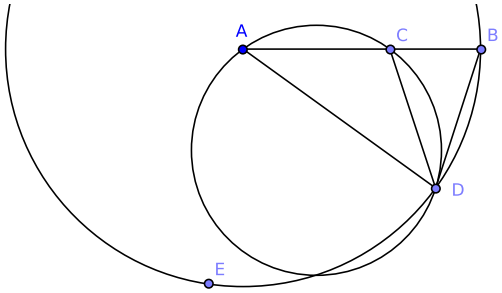


FIGURE 4.2.13. [4.10]

Construct $\overline{AB}$ and divide it at C such that the rectangle $\overline{AB} \cdot \overline{BC} = (\overline{AC})^2$ [2.6]. With A as center and $\overline{AB}$ as radius, construct $\bigcirc A$; on its circumference construct $\overline{BD} = \overline{AC}$ [4.1]. Also construct $\overline{AD}$. We claim that $\triangle ABD$ fulfills the required conditions.

Construct $\overline{CD}$ and the circle $\bigcirc ACD$ about $\triangle ACD$ [4.5].

Because the rectangle $\overline{AB} \cdot \overline{BC} = (\overline{AC})^2$ and $\overline{AC} = \overline{BD}$ (both by construction), it follows that $\overline{AB} \cdot \overline{BC} = (\overline{BD})^2$.

Since B is outside $\bigcirc ACD$, and from B the two segments $\overline{AB}$ and $\overline{BD}$ intersects $\bigcirc ACD$, and $\overline{AB}$ cuts $\bigcirc ACD$ while $\overline{BD}$ intersects $\bigcirc ACD$ at one point, and the rectangle $\overline{AB} \cdot \overline{BC} = (\overline{BD})^2$, by [3.32] $\overline{BD}$ touches $\bigcirc ACD$.

By [3.32], $\angle BDC = \angle DAC = \angle DAB$, and so

$$\begin{aligned} \angle BDC &= \angle DAB \\ \angle BDC + \angle CDA &= \angle DAB + \angle CDA \\ \angle BDA &= \angle DAB + \angle CDA \end{aligned}$$

Since $\angle DAB = \angle DAC$,

$$\begin{aligned} \angle BDA &= \angle DAC + \angle CDA \\ &= \angle BCD \end{aligned}$$

Since $\overline{AB} = \overline{AD}$ (each are radii of $\bigcirc A$), we have that $\angle BDA = \angle DBA$ by [1.5], from which it follows that

$$\angle BDA = \angle BCD = \angle CBD$$

By [1.6], $\overline{BD} = \overline{CD}$; but $\overline{BD} = \overline{AC}$ by construction, and so $\overline{AC} = \overline{CD}$. By [1.5], $\angle CDA = \angle DAB$. Since $\angle BDA = \angle CDA + \angle DAB$ by the above, we have that

$$\angle BDA = 2 \cdot \angle DAB$$

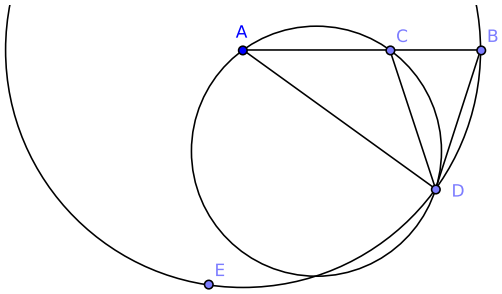


FIGURE 4.2.14. [4.10]

And since $\angle BDA = \angle DBA$,

$$\angle BDA = \angle DBA = 2 \cdot \angle DAB$$

This proves our claim. □

Exercises.

1. Prove that $\triangle ACD$ is an isosceles triangle whose vertical angle is equal to three times each of the base angles. [See the final chapter for a solution.]

2. Prove that $\overline{BD}$ is the side of a regular decagon inscribed in the circle $\bigcirc BDE$.

3. If $\overline{DB}$, $\overline{DE}$ and, $\overline{EF}$ are consecutive sides of a regular decagon inscribed in a circle, prove that $\overline{BF} - \overline{BD} =$ radius of a circle.

4. If E is the second point of intersection of the circle $\bigcirc ACD$ with $\bigcirc BDE$, then $\overline{DE} = \overline{DB}$. If $\overline{AE}$, $\overline{BE}$, $\overline{CE}$, and $\overline{DE}$ are constructed, then triangles $\triangle ACE$ and $\triangle ADE$ are each congruent with $\triangle ABD$.

5. $\overline{AC}$ is the side of a regular pentagon inscribed in the circle $\bigcirc ACD$, and $\overline{EB}$ the side of a regular pentagon inscribed in the circle $\bigcirc BDE$.

6. Since $\triangle ACE$ is an isosceles triangle, $(\overline{EB})^2 - (\overline{EA})^2 = \overline{AB} \cdot \overline{BC} = (\overline{BD})^2$; that is, the square of the side of a pentagon inscribed in a circle exceeds the square of the side of the decagon inscribed in the same circle by the square of the radius.

PROPOSITION 4.11. *INSCRIBING A REGULAR PENTAGON IN A GIVEN CIRCLE.*

It is possible to inscribe a regular pentagon in a given circle.

PROOF. We wish to inscribe a regular pentagon in $\bigcirc ABC$.

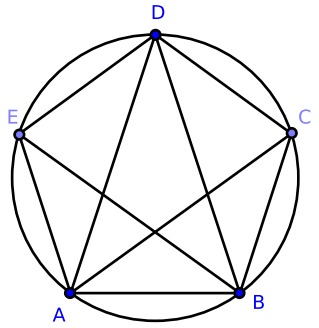


FIGURE 4.2.15. [4.11]

Construct any isosceles triangle having each base angle equal to double the vertical angle [4.10], and then construct $\triangle ABD$ equiangular to that triangle such that it is inscribed in $\bigcirc ABC$.

Bisect the angles $\angle DAB$ and $\angle ABD$ by constructing $\overline{AC}$ and $\overline{BE}$, respectively. Also construct $\overline{EA}$, $\overline{ED}$, $\overline{DC}$, and $\overline{CB}$. We claim that the figure $ABCDE$ is a regular pentagon.

Since $\angle DAB = \angle ABD = 2 \cdot \angle ADB$ by construction, $\overline{AC}$ bisects $\angle DAB$, and $\overline{BE}$ bisects $\angle ABD$, it follows that

$$\angle BAC = \angle CAD = \angle ADB = \angle DBE = \angle EBA$$

By [Cor. 3.29.1], the chords on which these angles stand are equal in length, so

$$\overline{AB} = \overline{BC} = \overline{CD} = \overline{DE} = \overline{EA}$$

Hence $ABCDE$ is equilateral.

Again, because the arcs AB and DE are equal in length, if we add the arc BCD to both, then the arc $ABCD$ is equal in length to the arc $BCDE$, and therefore the angles $\angle AED$, $\angle BAE$ which stand on them are equal [3.27].

Similarly, it can be shown that all of these angles are equal; therefore $ABCDE$ is equiangular. Hence, it is a regular pentagon; this proves our claim. □

Exercises.

1. Prove that the figure formed by the five diagonals of a regular pentagon is another regular pentagon.

2. If the alternate sides of a regular pentagon are extended to intersect, the five points of meeting form another regular pentagon.

3. Every two consecutive diagonals of a regular pentagon divide each other in the extreme and mean ratio.

4. Being given a side of a regular pentagon, construct it.

5. Divide a right angle into five equal parts.

PROPOSITION 4.12. *CIRCUMSCRIBING A REGULAR PENTAGON ABOUT A GIVEN CIRCLE.*

It is possible to circumscribe a regular pentagon about a given circle.

PROOF. We wish to construct a regular pentagon about $\bigcirc O$.

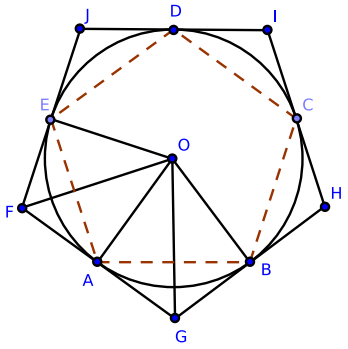


FIGURE 4.2.16. [4.12]

Use [4.11] to inscribe a regular pentagon inside $\bigcirc O$ with vertices at points A, B, C, D, and E; at these points, construct tangents $\overline{FG}$, $\overline{GH}$, $\overline{HI}$, $\overline{IJ}$, and $\overline{JF}$. We claim that $FGHIJ$ is the required circumscribed regular pentagon.

Construct $\overline{OE}$, $\overline{OA}$, and $\overline{OB}$. Because the angles $\angle OAF$ and $\angle OEF$ of the quadrilateral $AOEF$ are right angles by [3.18], the sum of the two remaining angles $\angle AOE + \angle AFE$ equals two right angles. Similarly, the sum $\angle AOB + \angle AGB$ equals two right angles; hence

$$\angle AOE + \angle AFE = \angle AOB + \angle AGB$$

But we also have that $\angle AOE = \angle AOB$ because they stand on equal arcs AE and AB [3.27]. Hence $\angle AFE = \angle AGB$. Similarly, the remaining angles of the figure $FGHIJ$ are equal, and so $FGHIJ$ is equiangular.

Now construct $\overline{OF}$ and $\overline{OG}$, and consider $\triangle EOF$ and $\triangle AOF$: they share equal sides $\overline{AF}$ and $\overline{FE}$ [3.17, #1], have side $\overline{FO}$ in common, and $\overline{AO} = \overline{EO}$. By [1.8], $\triangle EOF \cong \triangle AOF$, and so $\angle AFO = \angle EFO$; or, $\angle AFE$ is bisected at F.

Since $\angle AFE = \angle AFO + \angle EFO = 2 \cdot \angle AFO$, it follows that $\angle AFO = \frac{1}{2}\angle AFE$. Similarly, $\angle AGO = \frac{1}{2}\angle AGB$.

Consider $\triangle AFO$ and $\triangle AGO$: $\angle AFE = \angle AGB$, and so $\angle AFO = \angle AGO$; $\angle FAO = \angle GAO$ since each are right angles; finally, each shares side $\overline{AO}$. By [1.26], $\triangle AFO \cong \triangle AGO$, and so $\overline{AF} = \overline{AG}$.

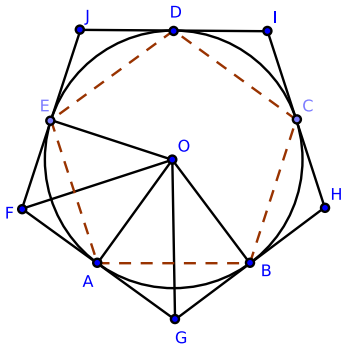


FIGURE 4.2.17. [4.12]

From this it follows that $\overline{GF} = 2 \cdot \overline{AF}$; similarly, $\overline{JF} = 2 \cdot \overline{EF}$. And since $\overline{AF} = \overline{EF}$, $\overline{GF} = \overline{JF}$, and so on for all remaining sides. Therefore, $FGHIJ$ is equilateral and equiangular; hence, it is a regular pentagon, which proves our claim. □

REMARK. This proposition is a particular case of the following general theorem (which has an analogous proof): "If tangents are constructed to a circle at the vertices of an inscribed polygon of any number of sides, they will form a regular polygon of the same number of sides circumscribed to the circle."

PROPOSITION 4.13. *INSCRIBING A CIRCLE IN A REGULAR PENTAGON.*

It is possible to inscribe a circle in a regular pentagon.

PROOF. We wish to inscribe $\bigcirc O$ in regular pentagon $ABCDE$.

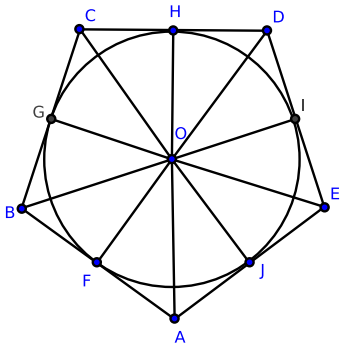


FIGURE 4.2.18. [4.13]

Bisect two adjacent angles $\angle JAF$ and $\angle FBG$ by constructing $\overline{AO}$ and $\overline{BO}$, respectively; we claim that the point of intersection of the bisectors, O, is the center of the required circle.

Construct $\overline{CO}$ as well as perpendiculars from O to the five sides of the pentagon.

Consider $\triangle ABO$ and $\triangle CBO$: $\overline{AB} = \overline{BC}$ by hypothesis, each shares side $\overline{BO}$, and $\angle ABO = \angle CBO$ by construction. By [1.4] $\triangle ABO \cong \triangle CBO$, and so$\angle BAO = \angle BCO$; however, $\angle BAO = \frac{1}{2} \cdot \angle BAE$ by construction. Therefore

$$\angle BCO = \frac{1}{2} \cdot \angle BAE = \frac{1}{2} \cdot \angle BCD$$

and hence $\overline{CO}$ bisects the angle $\angle BCD$.

Similarly, it may be proved that $\overline{DO}$ bisects the angle $\angle HDI$ and $\overline{EO}$ bisects the angle $\angle IEJ$.

Consider $\triangle BOF$ and $\triangle BOG$: $\angle OFB = \angle OGB$ since each are right, $\angle OBF = \angle OBG$ because $\overline{OB}$ bisects $\angle ABC$ by construction, and each shares side $\overline{OB}$. By [1.26], $\triangle BOF \cong \triangle BOG$, and so $\overline{OF} = \overline{OG}$.

Similarly, all the perpendiculars from O on the sides of the pentagon are equal. By [3.9], the circle whose center is O with radius $\overline{OF}$ is inscribed in regular pentagon $ABCDE$. □

PROPOSITION 4.14. *CIRCUMSCRIBING A CIRCLE ABOUT A REGULAR PENTAGON.*

It is possible to circumscribe a circle about a regular pentagon.

PROOF. We wish to construct $\bigcirc O$ about regular pentagon $ABCDE$.

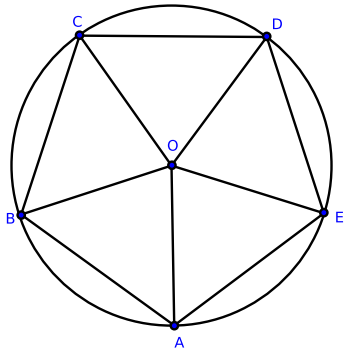


FIGURE 4.2.19. [4.14]

Bisect adjacent angles $\angle BAE$ by $\overline{AO}$ and $\angle ABC$ by $\overline{BO}$. We claim that O, the point of intersection of the bisectors, is the center of the required circle.

Construct $\overline{OC}$, $\overline{OD}$, and $\overline{OE}$. Consider $\triangle ABO$ and $\triangle CBO$: $\overline{AB} = \overline{BC}$ by hypothesis, $\overline{BO}$ in common, and $\angle ABO = \angle CBO$ by construction. By [1.4], $\triangle ABO \cong \triangle CBO$, and so $\angle BAO = \angle BCO$.

But $\angle BAE = \angle BCD$ since $ABCDE$ is a regular pentagon. Since $\angle BAO = \frac{1}{2}\angle BAE$ by construction, $\angle BCO = \frac{1}{2}\angle BCD$. Hence, $\overline{CO}$ bisects $\angle BCD$.

Similarly, it may be proved that $\overline{DO}$ bisects $\angle CDE$ and $\overline{EO}$ bisects the angle $\angle DEA$.

Because $\angle EAB = \angle ABC$, it follows that $\angle OAB = \angle OBA$. Consider $\triangle OBA$: by [1.4], $\overline{OA} = \overline{OB}$. Similarly, we can show that

$$\overline{OA} = \overline{OB} = \overline{OC} = \overline{OD} = \overline{OE}$$

By [3.9], O is the center of a circle with radius $\overline{OA}$ which passes through points B, C, D, and E, and is constructed about the regular pentagon $ABCDE$. This proves our claim. □

PROPOSITION 4.15. *INSCRIBING A REGULAR HEXAGON IN A CIRCLE.*

It is possible to inscribe a regular hexagon in a circle.

PROOF. We wish to to inscribe regular hexagon $ABCDEF$ in $\bigcirc O$.

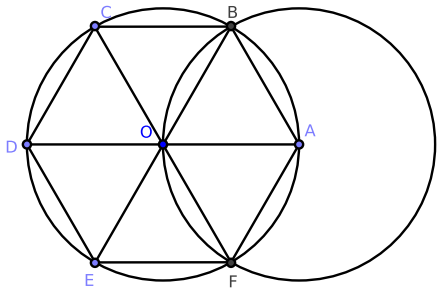


FIGURE 4.2.20. [4.15]

Take any point A on the circumference of $\bigcirc O$ and construct $\overline{AO}$. With A as center and $\overline{AO}$ as radius, construct the circle $\bigcirc A$, intersecting $\bigcirc O$ at the points B and F.

Construct $\overline{OB}$ and $\overline{OF}$; extend $\overline{AO}$ to intersect $\bigcirc A$ at D, extend $\overline{BO}$ to intersect $\bigcirc A$ at E, and extend $\overline{FO}$ to intersect $\bigcirc A$ at C. Also construct $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$; we claim that hexagon $ABCDEF$ is the required hexagon.

Notice that $\overline{OA} = \overline{OB}$ since each are radii of $\bigcirc O$. Similarly, $\overline{AB} = \overline{OA}$ since each are radii of $\bigcirc A$. Hence,

$$\overline{OA} = \overline{OB} = \overline{AB}$$

and so $\triangle OAB$ is equilateral.

Since the sum of the interior angles of a triangle is two right angles and equilateral triangles have equal angles,

$$\angle AOB = \angle OBA = \angle OAB$$

Mutatis mutandis, we can show that

$$\angle AOB = \angle AOF = \angle FOE = \angle EOD = \angle DOC = \angle BOC$$

By [Cor. 3.29.1],

$$\overline{AB} = \overline{BC} = \overline{CD} = \overline{DE} = \overline{EF} = \overline{FA}$$

and so hexagon $ABCDEF$ is equilateral.

Also

$$\overline{OA} = \overline{OB} = \overline{OC} = \overline{OD} = \overline{OE} = \overline{OF}$$

since each are radii of $\bigcirc O$. By [1.8], each sub-triangle of hexagon $ABCDEF$ is congruent. It follows that

$$\angle ABC = \angle BCD = \angle CDE = \angle DEF = \angle EFA = \angle FAB$$

and so $ABCDEF$ is equiangular.

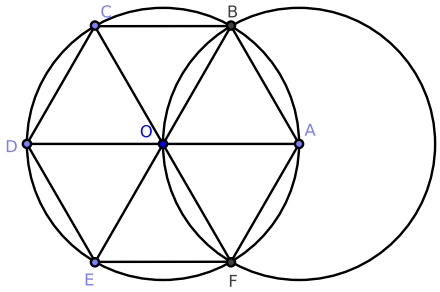


FIGURE 4.2.21. [4.15]

This proves our claim. □

REMARK. [4.13] and [4.14] are particular cases of the following theorem: "A regular polygon of any number of sides has one circle inscribed in it and another constructed about it, and both circles are concentric."

COROLLARY. *4.15.1. The length of the side of a regular hexagon inscribed in a circle is equal to the circle's radius.*

COROLLARY. *4.15.2. If three alternate angles of a hexagon are joined, they form an inscribed equilateral triangle.*

Exercises.

1. The area of a regular hexagon inscribed in a circle is equal to twice the area of an equilateral triangle inscribed in the circle. Also, the square of the side of the triangle equals three times the square of the area of the side of the hexagon.

2. If the diameter of a circle is extended to C until the extended segment is equal to the radius, then the two tangents from C and their chord of contact form an equilateral triangle.

3. The area of a regular hexagon inscribed in a circle is half the area of an equilateral triangle and three-fourths of the area of a regular hexagon circumscribed to the circle.

4. Prove [Cor. 4.15.1].

5. Prove [Cor. 4.15.2].

PROPOSITION 4.16. *INSCRIBING A REGULAR FIFTEEN-SIDED POLYGON IN A GIVEN CIRCLE.*

It is possible to inscribe a regular, fifteen-sided polygon in a given circle.

PROOF. We wish to inscribe a regular fifteen-sided polygon in $\bigcirc O$.

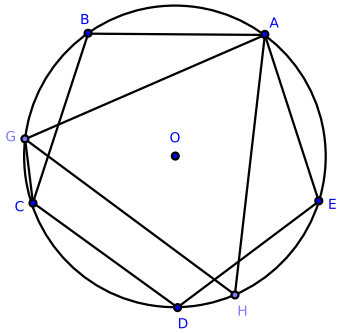


FIGURE 4.2.22. [4.16]

Inscribe a regular pentagon $ABCDE$ in the circle $\bigcirc O$ [4.11] and also an equilateral triangle $\triangle AGH$ [4.2]. Construct $\overline{CG}$. We claim that $\overline{CG}$ is a side of the required polygon.

Since $ABCDE$ is a regular pentagon, the arc ABC is $\frac{2}{5}$ of the circumference.

Since $\triangle AGH$ is an equilateral triangle, the arc ABG is $\frac{1}{3}$ of the circumference.

Hence the arc $\overline{GC}$, which is the difference between these two arcs, is equal to $\frac{2}{5} - \frac{1}{3} = \frac{1}{15}$ of the entire circumference.

Therefore, if chords equal in length to $\overline{GC}$ are placed round the circle [4.1], we have a regular fifteen-sided polygon (i.e., a quindecagon) inscribed in it. □

REMARK. Until 1801, no regular polygon could be constructed by lines and circles only except those discussed in this Book IV of Euclid and those obtained from them by the continued bisection of the arcs of which their sides are the chords; but in that year, Gauss proved that if $2n + 1$ is a prime number, then regular polygons of $2n + 1$ sides are constructible by elementary geometry.

Exam questions for chapter 4.

1. What is the subject-matter of chapter 4?

2. When is one polygon said to be inscribed in another?

3. When is one polygon said to be circumscribed about another?

4. When is a circle said to be inscribed in a polygon?

5. When is a circle said to be circumscribed about a polygon?

6. What is meant by reciprocal propositions? (Ans. In reciprocal propositions, to every line in one there corresponds a point in the other; and, conversely, to every point in one there corresponds a line in the other.)

7. Give instances of reciprocal propositions in chapter 4.

8. What is a regular polygon?

9. What figures can be inscribed in, and circumscribed about, a circle by means of chapter 4?

10. What regular polygons has Gauss proved to be constructible by the line and circle?

11. What is meant by escribed circles?

12. How many circles can be constructed to touch three lines forming a triangle?

13. What is the centroid of a triangle?

14. What is the orthocenter?

15. What is the circumcenter?

16. What is the polar circle?

17. What is the "nine-points circle"?

18. How does a nine-points circle get its name?

19. Name the nine points that a nine-points circle passes through.

20. What three regular figures can be used in filling up the space round a point? (Ans. Equilateral triangles, squares, and hexagons.)

21. If the sides of a triangle are 13, 14, 15 units in length, what are the values of the radii of its inscribed and escribed circles?

22. What is the radius of the circumscribed circle?

23. What is the radius of its nine-points circle?

24. What is the distance between the centers of its inscribed and circumscribed circles?

25. If r is the radius of a circle, what is the area:

(a) of its inscribed equilateral triangle?

(b) of its inscribed square?

(c) its inscribed pentagon?

(d) its inscribed hexagon?

(e) its inscribed octagon?

(f) its inscribed decagon?

26. With the same hypothesis, find the sides of the same regular figures.

Exercises for chapter 4.

1. If a circumscribed polygon is regular, prove that the corresponding inscribed polygon is also regular. Also prove the converse.

2. If a circumscribed triangle is isosceles, the corresponding inscribed triangle is isosceles. Also prove the converse.

3. If the two isosceles triangles in #2 have equal vertical angles, they are both equilateral.

4. Divide an angle of an equilateral triangle into five equal parts.

5. Inscribe a circle in a sector of a given circle.

6. Suppose that segments $\overline{DE}$, $\overline{BC}$ of $\triangle ABC$ are parallel: $\overline{DE} \parallel \overline{BC}$. Prove that the circles constructed about the triangles $\triangle ABC$, $\triangle ADE$ touch at A.

7. If the diagonals of a cyclic quadrilateral intersect at E, prove that the tangent at E to the circle about the triangle $\triangle ABE$ is parallel to $\overline{CD}$.

8. Inscribe a regular octagon in a given square.

9. If a segment of given length slides between two given lines, find the locus of the intersection of perpendiculars from its endpoints to the given lines.

10. If the perpendicular to any side of a triangle at its midpoint meets the internal and external bisectors of the opposite angle at the points D and E, prove that D, E are points on the circumscribed circle.

11. Through a given point P, construct a chord of a circle so that the intercept $\overline{EF}$ stands opposite a given angle at point X.

12. In a given circle, inscribe a triangle having two sides passing through two given points and the third parallel to a given line.

13. Given four points, no three of which are collinear, construct a circle which is equidistant from them.

14. In a given circle, inscribe a triangle whose three sides pass through three given points.

15. Construct a triangle, being given:

(a) the radius of the inscribed circle, the vertical angle, and the perpendicular from the vertical angle on the base.

(b) the base, the sum or difference of the other sides, and the radius of the inscribed circle, or of one of the escribed circles.

(c) the centers of the escribed circles.

16. If F is the midpoint of the base of a triangle, DE the diameter of the circumscribed circle which passes through F, and L the point where a parallel to the base through the vertex meets $\overline{DE}$, prove that $\overline{DL} \cdot \overline{FE}$ is equal to the square of half the sum and $\overline{DF} \cdot \overline{LE}$ is equal to the square of half the difference of the two remaining sides.

17. If from any point within a regular polygon of n sides perpendiculars fall on the sides, their sum is equal to n times the radius of the inscribed circle.

18. The sum of the perpendiculars falling from the vertices of a regular polygon of n sides on any line is equal to n times the perpendicular from the center of the polygon on the same line.

19. If R denotes the radius of the circle circumscribed about a triangle $\triangle ABC$, r, r', r'', r''' are the radii of its inscribed and escribed circles; δ, δ', δ'' are the perpendiculars from its circumcenter on the sides; μ, μ', μ'' are the segments of these perpendiculars between the sides and circumference of the circumscribed circle, then we have the equalities:

$$\begin{aligned} r' + r'' + r''' &= 4R + r \quad (1) \\ \mu + \mu' + \mu'' &= 2R - r \quad (2) \\ \delta + \delta' + \delta'' &= R + r \quad (3) \end{aligned}$$

The relation (3) supposes that the circumcenter is inside the triangle.

20. Take a point D from the side BC of a triangle $\triangle ABC$ and suppose we construct a transversal EDF through it; suppose we also construct circles about

the triangles $\triangle DBF$, $\triangle ECD$. The locus of their second point of intersection is a circle.

21. In every quadrilateral circumscribed about a circle, the midpoints of its diagonals and the center of the circle are collinear.

22. Find on a given line a point P, the sum or difference of whose distances from two given points may be given.

23. Find a point such that, if perpendiculars fall from it on four given lines, their feet are collinear.

24. The line joining the orthocenter of a triangle to any point P in the circumference of its circumscribed circle is bisected by the line of co-linearity of perpendiculars from P on the sides of the triangle.

25. The orthocenters of the four triangles formed by any four lines are collinear.

26. If a semicircle and its diameter are touched by any circle either internally or externally, then twice the rectangle contained by the radius of the semicircle and the radius of the tangential circle is equal to the rectangle contained by the segments of any secant to the semicircle through the point of intersection of the diameter and touching circle.

27. If ρ, ρ' are radii of two circles touching each other at the center of the inscribed circle of a triangle where each touches the circumscribed circle, prove that

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{2}{r}$$

and state and prove corresponding theorems for the escribed circles.

28. If from any point in the circumference of the circle, circumscribed about a regular polygon of n sides, segments are constructed to its vertices, the sum of their squares is equal to $2n$ times the square of the radius.

29. In the same case as the above, if the lines are constructed from any point in the circumference of the inscribed circle, prove that the sum of their squares is equal to n times the sum of the squares of the radii of the inscribed and the circumscribed circles.

30. State the corresponding theorem for the sum of the squares of the lines constructed from any point in the circumference of any concentric circle.

31. If from any point in the circumference of any concentric circle perpendiculars are let fall on all the sides of any regular polygon, the sum of their squares is constant.

32. See #31. For the inscribed circle, the constant is equal to $3n/2$ times the square of the radius.

33. See #31. For the circumscribed circle, the constant is equal to n times the square of the radius of the inscribed circle, together with $\frac{1}{2}n$ times the square of the radius of the circumscribed circle.

34. If the circumference of a circle whose radius is R is divided into seventeen equal parts and AO is the diameter constructed from one of the points of division (A), and if ρ_1, ρ_2, ..., ρ_8 denote the chords from O to the points of division, A_1, A_2, ..., A_8 on one side of AO, then

$$\rho_1\rho_2\rho_4\rho_8 = \rho_3\rho_5\rho_6\rho_7 = R^4$$

Let the supplemental chords corresponding to ρ_1, ρ^2, etc., be written as r_1, r_2, etc. Then by [3.35, #2], we have that

$$\begin{aligned} \rho_1 r_1 &= Rr_2 \\ \rho_2 r_2 &= Rr_4 \\ \rho_4 r_4 &= Rr_8 \\ \rho_8 r_8 &= Rr_1 \end{aligned}$$

Hence, $\rho_1\rho_2\rho_4\rho_8 = R^4$.

And it may be similarly proved that

$$\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6\rho_7\rho_8 = R^8$$

Therefore, $\rho_3\rho_5\rho_6\rho_7 = R^4$.

35. If from the midpoint of the segment joining any two of four concyclic points a perpendicular falls on the line joining the remaining two, the six perpendiculars thus obtained are concurrent.

36. The greater the number of sides of a regular polygon circumscribed about a given circle, the less will be its perimeter.

37. The area of any regular polygon of more than four sides circumscribed about a circle is less than the square of the diameter.

38. Four concyclic points taken three by three determine four triangles, the centers of whose nine-points circles are concyclic.

39. If two sides of a triangle are given in position, and if their included angle is equal to an angle of an equilateral triangle, the locus of the center of its nine-points circle is a straight line.

40. If in the hypothesis and notation of #34, α, β denote any two suffixes whose sum is less than 8 where $\alpha > \beta$, then

$$\rho_\alpha \rho_\beta = R(\rho_{\alpha-\beta} + \rho_{\alpha+\beta})$$

For example, $\rho_1 \rho_4 = R(\rho_3 + \rho_5)$ [3.35, #7].

In the same case, if the suffixes be greater than 8, then

$$\rho_\alpha \cdot \rho_\beta = R(\rho_{\alpha-\beta} - \rho_{17-\alpha-\beta})$$

For instance, $\rho_8 \cdot \rho_2 = R(\rho_6 - \rho_7)$ [3.35, #6].

41. Two lines are given in position. Construct a transversal through a given point, forming with the given lines a triangle of given perimeter.

42. Given the vertical angle and perimeter of a triangle, construct it with any of the following data:

(a) The bisector of the vertical angle;

(b) the perpendicular from the vertical angle on the base;

(c) the radius of the inscribed circle.

43. In a given circle, inscribe a triangle so that two sides may pass through two given points and that the third side may be a maximum or a minimum.

44. If s is the semi-perimeter of a triangle, and if r', r'', r''' are the radii of its escribed circles, then

$$r' \cdot r'' + r'' \cdot r''' + r''' \cdot r' = s^2$$

45. The feet of the perpendiculars from the endpoints of the base on either bisector of the vertical angle, the midpoint of the base, and the foot of the perpendicular from the vertical angle on the base are concyclic.

46. Given the base of a triangle and the vertical angle, find the locus of the center of the circle passing through the centers of the escribed circles.

47. The perpendiculars from the centers of the escribed circles of a triangle on the corresponding sides are concurrent.

48. If $\overline{AB}$ is the diameter of a circle, $\overline{PQ}$ is any chord cutting $\overline{AB}$ at O, and if the segments $\overline{AP}$, $\overline{AQ}$ intersect the perpendicular to $\overline{AB}$ at O (at D and E respectively), then the points A, B, D, E are concyclic.

49. If the sides of a triangle are in arithmetical progression, and if R, r are the radii of the circumscribed and inscribed circles, then $6Rr$ is equal to the rectangle contained by the greatest and least sides.

50. Inscribe in a given circle a triangle having its three sides parallel to three given lines.

51. If the sides $\overline{AB}$, $\overline{BC}$, etc., of a regular pentagon is bisected at the points A', B', C', D', E', and if the two pairs of alternate sides $\overline{BC}$, $\overline{AE}$ and $\overline{AB}$, $\overline{DE}$ meet at the points A'', E'', respectively, prove that

$$\triangle A''AE'' - \triangle A'AE' = \text{pentagon } A'B'C'D'E'$$

52. In a circle, prove that an equilateral inscribed polygon is regular; also prove that if the number of its sides are odd, then it is an equilateral circumscribed polygon.

53. Prove that an equiangular circumscribed polygon is regular; also prove that if the number of its sides are odd, then it is an equilateral inscribed polygon.

54. The sum of the perpendiculars constructed to the sides of an equiangular polygon from any point inside the figure is constant.

55. Express the sides of a triangle in terms of the radii of its escribed circles.

CHAPTER 5

Theory of Proportions

Chapter 5, like Chapter 2, proves a number of algebraic propositions that are more familiar to us in the form of equations. Algebra as we know it had not been developed when Euclid wrote "The Elements", and so the results are more of historical importance than practical use (except when they are used in subsequent proofs). This is why Book V appears here in truncated form.

5.1. Definitions

0. Variables a, b, c, x, y, ... represent positive real numbers.

1. Suppose that x and y are two positive integers where $x < y$. We say that x is a *factor* of y when there exists a positive integer $n > 1$ such that $nx = y$.

2. Suppose the above. We also say that y is a *multiple* of x.

3. Suppose that x and y are two positive integers such that $x \neq 0$. A *ratio* is the number $\frac{y}{x}$.

4. Numbers are said to have a *ratio* to one another when the lesser number can be multiplied so as to exceed the greater.

5. Multiplying by positive numbers preserve both equalities and inequalities.

6. Numbers which have the same ratio are called *proportions*. When four numbers are proportions, it may be described as: "The first is to the second as the third is to the fourth." Or:

$$\frac{a}{b} = \frac{c}{d}$$

As a ratio, the above equality may be written as:

$$a : b = c : d$$

7. Inequalities of fractions:

$$\frac{a}{b} > \frac{c}{d} \iff \frac{ad}{bd} > \frac{bc}{bd}$$

The symbol $>$ can be replaced with $\geq$, $<$, and $\leq$.

8. Numbers which have the same ratio are called *proportional*.

9. Proportions consist of at minimum three terms.

REMARK. This definition has the same fault as some of the others: it is not a definition, it's an inference. It occurs when the means in a proportion are equal, and so there are four terms. As an example, take the numbers 4, 6, 9. Here the ratio of $4 : 6$ is $2/3$, and the ratio of $6 : 9$ is $2/3$; therefore 4, 6, 9 are continued proportionals. But, in reality, there are four terms: the full proportion is $4 : 6 = 6 : 9$.

10. When three numbers are continued proportions, the first is said to have to the third the *duplicate ratio* of that which it has to the second.

Algebraically,

$$\begin{aligned} \frac{x}{y} &= \frac{y}{z} \\ \left(\frac{x}{y}\right)^2 &= \frac{x}{y} \cdot \frac{y}{z} \\ \left(\frac{x}{y}\right)^2 &= \frac{x}{z} \end{aligned}$$

Example: $4 : 25$ *is the duplicate ratio of* $2 : 5$.

11. When four numbers are continual proportionals, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second.

Algebraically,

$$\frac{u}{v} = \frac{v}{w} = \frac{w}{x}$$

so

$$\begin{aligned} \left(\frac{u}{v}\right)^3 &= \frac{u}{v} \cdot \frac{v}{w} \cdot \frac{w}{x} \\ \left(\frac{u}{v}\right)^3 &= \frac{u}{x} \end{aligned}$$

Example: $27 : 8$ *is the triplicate ratio of* $3 : 2$.

12. When we have numbers of the same kind greater than two, the first is said to have to the last the ratio compounded of the ratios of the first to the second, of the second to the third, of the third to the fourth, etc.

13. A ratio whose antecedent is greater than its consequent is called a ratio of greater inequality. A ratio whose antecedent is less than its consequent is called a ratio of lesser inequality.

14. *Harmonic division* of a segment $\overline{AB}$ means identifying two points C and D such that $\overline{AB}$ is divided internally and externally in the same ratio $\frac{CA}{CB} = \frac{DA}{DB}$.

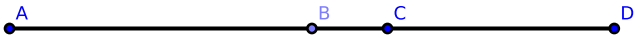


FIGURE 5.1.1. Here, the ratio is 2. Specifically, the distance $\overline{AC}$ is one unit, the distance $\overline{CB}$ is half a unit, the distance $\overline{AD}$ is three units, and the distance $\overline{BD}$ is 1.5 units.

Harmonic division of a line segment is reciprocal: if points C and D divide the segment $\overline{AB}$ harmonically, the points A and B also divide the line segment $\overline{CD}$ harmonically. In that case, the ratio is given by $\frac{BC}{BD} = \frac{AC}{AD}$ which equals one-third in the example above. (Note that the two ratios are not equal.)[1]

5.2. Propositions from Book V

PROPOSITION 5.1. *If any number of numbers are each the same multiple of the same number of other numbers, then the sum is that multiple of the sum.*

COROLLARY. *5.1.1. [5.1] is equivalent to* $kx + ky = k(x + y)$.

PROPOSITION 5.2. *If a first number is the same multiple of a second that a third is of a fourth, and a fifth also is the same multiple of the second that a sixth is of the fourth, then the sum of the first and fifth also is the same multiple of the second that the sum of the third and sixth is of the fourth.*

COROLLARY. *5.2.1. [5.2] is equivalent to the following: if* $kv = x$, $kw = r$, $mv = y$, *and* $mw = u$, *then* $x + y = (k + m)v$ *and* $r + u = (k + m)w$.

[1] http://en.wikipedia.org/wiki/Harmonic_division

PROPOSITION 5.3. *If a first number is the same multiple of a second that a third is of a fourth, and if equimultiples are taken of the first and third, then the numbers taken also are equimultiples respectively, the one of the second and the other of the fourth.*

COROLLARY. *5.3.1. [5.3] is equivalent to the following: Let* $A = kB$ *and* $C = kD$*. If* $EF = mA$ *and* $GM = mC$*, then* $EF = mkB$ *and* $GH = mkD$*.*

PROPOSITION 5.4. *If a first number has to a second the same ratio as a third to a fourth, then any equimultiples whatever of the first and third also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.*

COROLLARY. *5.4.1. [5.4] is equivalent to: if* $\frac{A}{B} = k = \frac{C}{D}$*, then* $A = kB$ *and* $C = kD$*.*

PROPOSITION 5.5. *If a number is the same multiple of a number that a subtracted part is of a subtracted part, then the remainder also is the same multiple of the remainder that the whole is of the whole.*

COROLLARY. *5.5.1. [5.5] is equivalent to: if* $x + y = k(m + n)$ *and* $x = km$*, then* $y = kn$*.*

PROPOSITION 5.6. *If two numbers are equimultiples of two numbers, and any numbers subtracted from them are equimultiples of the same, then the remainders either equal the same or are equimultiples of them.*

COROLLARY. *5.6.1. [5.6] is equivalent to: if* $x + y = km$*,* $u + v = kn$*,* $x = lm$*,* $y = ln$*, and all variables are positive, then* $y = (k-l)m$ *and* $v = (k-l)n$ *whenever* $k > l$*.*

PROPOSITION 5.7. *Equal numbers have to the same the same ratio; and the same has to equal numbers the same ratio.*

COROLLARY. *5.7.1. If any numbers are proportional, then they are also proportional inversely.*

COROLLARY. *5.7.2. [5.7] is equivalent to: if* $a = b$*, then* $a : c = b : c$*, and* $c : a = c : b$*.*

PROPOSITION 5.8. *Of unequal numbers, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.*

COROLLARY. *5.8.1. [5.8] is equivalent to: if* $AB > C$ *and* $D > 0$*, then* $AB = C + k$ *where* $k > 0$*, and* $\frac{AB}{D} = \frac{C+k}{D} = \frac{C}{D} + \frac{k}{D} > \frac{C}{D}$*. It follows that* $\frac{D}{C} > \frac{D}{AB}$*, since all quantities are positive.*

PROPOSITION 5.9. *Numbers which have the same ratio to the same equal one another; and numbers to which the same has the same ratio are equal.*

COROLLARY. *5.9.1. [5.9] is equivalent to: if* $A = kC$ *and* $B = kC$*, then* $A = B$*.*

PROPOSITION 5.10. *Of numbers which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.*

COROLLARY. *5.10.1. [5.10] is equivalent to: if* $\frac{A}{C} > \frac{B}{C}$ *and* $C > 0$*, then* $A > B$*.*

PROPOSITION 5.11. *Ratios which are the same with the same ratio are also the same with one another.*

COROLLARY. *5.11.1. [5.11] is equivalent to: if* $\frac{A}{B} = \frac{C}{D}$ *and* $\frac{C}{D} = \frac{E}{F}$*, then* $\frac{A}{B} = \frac{E}{F}$*. This is the transitive property for fractions.*

PROPOSITION 5.12. *If any number of numbers are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents.*

COROLLARY. *5.12.1. [5.12] is equivalent to: if* $A = kB$*,* $C = kD$*,* $E = kF$*, then* $A + C + E = kB + kD + kF = k(B + D + F)$*.*

PROPOSITION 5.13. *If a first number has to a second the same ratio as a third to a fourth, and the third has to the fourth a greater ratio than a fifth has to a sixth, then the first also has to the second a greater ratio than the fifth to the sixth.*

COROLLARY. *5.13.1. [5.13] is equivalent to: if* $A = kB$, $C = kD$, $C = lD$, $E = jF$ *and* $l > j$, *then* $k = l$ *and so* $k > j$.

PROPOSITION 5.14. *If a first number has to a second the same ratio as a third has to a fourth, and the first is greater than the third, then the second is also greater than the fourth; if equal, equal; and if less, less.*

COROLLARY. *5.14.1. [5.14] is equivalent to: if* $A = kB$, $C = kD$, $A > C$, *and* $k > 0$, *then* $kB = A > C = kD$ *and so* $B > D$. *If* $A < C$ *and* $B < D$, *the result follows mutatis mutandis.*

PROPOSITION 5.15. *Parts have the same ratio as their equimultiples.*

COROLLARY. *5.15.1. [5.15] is equivalent to: if* $AB = kC$, $DE = kF$, *and* $C = mF$, *then* $AB = kmF = mDE$.

PROPOSITION 5.16. *If four numbers are proportional, then they are also proportional alternately.*

COROLLARY. *5.16.1. [5.16] is equivalent to: if* $\frac{A}{B} = \frac{C}{D}$, *then* $\frac{A}{C} = \frac{B}{D}$.

PROPOSITION 5.17. *If numbers are proportional taken jointly, then they are also proportional taken separately.*

COROLLARY. *5.17.1. [5.17] is equivalent to: if* $x + y = ky$, $u + v = kv$, *and* $x = ly$, *then* $ly + y = ky$, *or* $l + 1 = k$. *Thus* $u + v = (l + 1)v$, *or* $u = lv$.

PROPOSITION 5.18. *If numbers are proportional taken separately, then they are also proportional taken jointly.*

COROLLARY. *5.18.1. [5.18] is equivalent to: if* $x = ky$, $u = kv$, *and* $x + y = ly$, *then* $(k + 1)y = ly$ *and so* $k + 1 = l$ *and* $u + v = kv = (k + 1)v = lv$.

PROPOSITION 5.19. *If a whole is to a whole as a part subtracted is to a part subtracted, then the remainder is also to the remainder as the whole is to the whole.*

COROLLARY. *5.19.1. If numbers are proportional taken jointly, then they are also proportional in conversion.*

COROLLARY. *5.19.2. [5.19] is equivalent to: if* $x+y = k(u+v)$ *and* $x = ku$, *then* $y = kv$.

REMARK. 5.19.3. If $u : v$, the [5.16] [5.17], [5.18], and [5.19] show the following proportions to be equivalent[2]:

(1) $u : v = x : y$
(2) $(u + v) : v = (x + y) : y$
(3) $(u + v) : u = (x + y) : x$
(4) $(u + v) : (x + y) = v : y$
(5) $(u + v) : (x + y) = u : x$
(6) $u : x = v : y$

(2)-(5) also hold when $+$ is replaced by $-$.

PROPOSITION 5.20. *If there are three numbers, and others equal to them in multitude, which taken two and two are in the same ratio, and if the first is greater than the third, then the fourth is also greater than the sixth; if equal, equal, and; if less, less.*

COROLLARY. *5.20.1. [5.20] is equivalent to: let* $A = kB$, $B = lC$, $D = kE$, $E = lF$, *and* $A > C$. *We wish to show that* $D > F$.

Suppose that $A = c + m$, $m > 0$. *Then* $A = klC$, $D = klF$, *and so* $\frac{A}{C} = \frac{D}{F}$.

Now $\frac{A}{C} > 1$ *since* $A > C$. *If* $D = F$, $\frac{A}{C} = 1$; *and if* $D < F$, $\frac{A}{C} < 1$. *Hence,* $D > F$.

The remaining cases follow mutandis mutatis.

[2]David E. Joyce provides these results at:
aleph0.clarku.edu/~djoyce/java/elements/bookX/propX29.html

PROPOSITION 5.21. *If there are three numbers, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them is perturbed, then, if the first number is greater than the third, then the fourth is also greater than the sixth; if equal, equal; and if less, less.*

COROLLARY. *5.21.1. The result of [5.21] is the same as the result [5.20].*

PROPOSITION 5.22. *If there are any number of numbers whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, then they are also in the same ratio.*

COROLLARY. *5.22.1. [5.22] is equivalent to: if* $A = kB$, $B = lC$, $D = kE$, *and* $E = lF$, *then* $A = klC$ *and* $D = klF$.

PROPOSITION 5.23. *If there are three numbers, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then they are also in the same ratio.*

COROLLARY. *5.23.1. The result of [5.23] is the same as the result of [5.22].*

PROPOSITION 5.24. *If a first number has to a second the same ratio as a third has to a fourth, and also a fifth has to the second the same ratio as a sixth to the fourth, then the sum of the first and fifth has to the second the same ratio as the sum of the third and sixth has to the fourth.*

COROLLARY. *5.24.1. [5.24] is equivalent to: if* $x = km$, $u = kn$, $y = lm$, *and* $v = ln$, *then* $x + y = km + lm = (k + l)m$ *and* $u + v = kn + ln = (k + l)n$.

PROPOSITION 5.25. *If four numbers are proportional, then the sum of the greatest and the least is greater than the sum of the remaining two.*

COROLLARY. *5.25.1. [5.25] is equivalent to: let* $x + y = k(u + v)$, $k > 1$, *and* $x = ku$. *Since* $x + y = ku + kv$, $y = kv$; *and since* $k > 1$, $y > v$.

Exam questions for chapter 5.

1. What is the subject-matter of this chapter?

2. When is one number said to be a multiple of another?

3. What is a sub-multiple or measure?

4. What are equimultiples?

5. What is the ratio of two commensurable numbers?

6. What is meant by the ratio of incommensurable numbers?

7. Give an illustration of the ratio of incommensurables.

8. What are the terms of a ratio called?

9. What is duplicate ratio?

10. Define triplicate ratio.

11. What is proportion? (Ans. Equality of ratios.)

12. How many ratios in a proportion?

13. When is a segment divided harmonically?

14. What are reciprocal ratios?

Chapter 5 exercises.

1. A ratio of greater inequality is increased by diminishing its terms by the same quantity, and diminished by increasing its terms by the same quantity.

2. A ratio of lesser inequality is diminished by diminishing its terms by the same quantity, and increased by increasing its terms by the same quantity.

3. If four numbers are proportionals, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.

4. If two sets of four numbers are proportionals, and if we multiply corresponding terms together, the products are proportionals.

5. If two sets of four numbers are proportionals, and if we divide corresponding terms, the quotients are proportionals.

6. If four numbers are proportionals, their squares, cubes, etc., are proportionals. [See the final chapter for a solution.]

7. If two proportions have three terms of one respectively equal to three corresponding terms of the other, the remaining term of the first is equal to the remaining term of the second.

8. If three numbers are continual proportionals, the first is to the third as the square of the difference between the first and second is to the square of the difference between the second and third.

9. If $\overline{AB}$ is cut harmonically at C and D and is bisected at O, prove that $\overline{OC}$, $\overline{OB}$, $\overline{OD}$ are continual proportionals.

10. In the same case, if O' is the midpoint of $\overline{CD}$, prove that $(\overline{OO'})^2 = (\overline{OB})^2 + (\overline{OD})^2$.

11. Continuing from #10, show that $\overline{AB} \cdot (\overline{AC} + \overline{AD}) = 2 \cdot \overline{AC} \cdot \overline{AD}$, or $\frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB}$

12. Continuing from #10, show that $\overline{CD} \cdot (\overline{AD} + \overline{BD}) = 2 \cdot \overline{AD} \cdot \overline{BD}$, or $\frac{1}{BD} + \frac{1}{AD} = \frac{2}{AC}$

13. Continuing from #10, show that $\overline{AB} \cdot \overline{CD} = 2 \cdot \overline{AD} \cdot \overline{CB}$.

CHAPTER 6

Applications of Proportions

If we wish to state that the area of $\triangle ABC$ divided by the area of $\triangle DEF$ is equal to the area of $\triangle GHI$ divided by the area of $\triangle JKL$, we may write either

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\triangle GHI}{\triangle JKL}$$

or

$$\triangle ABC : \triangle DEF = \triangle GHI : \triangle JKL$$

This is comparable to how we write $\triangle GHI = \triangle JKL$ to indicate that the area of $\triangle GHI$ is equal to the area of $\triangle JKL$.

6.1. Definitions

1. *Similar polygons* are polygons that both have the same shape, or one has the same shape as the mirror image of the other. More precisely, one polygon can be obtained from the other by uniformly scaling (enlarging or shrinking), possibly with additional translation, rotation and reflection. This means that either object can be rescaled, repositioned, and reflected, so as to coincide precisely with the other object.

A modern perspective of similarity is to consider polygons similar if one appears congruent to the other when zoomed in or out at some level.[1]

Similar polygons agree in shape; if they also agree in size, then they are congruent. If polygons A and B are similar, we will denote this as $A \sim B$.

(a) When the shape of a figure is given, it is said to be given in species. Thus a triangle whose angles are given is given in species. Hence, similar figures are of the same species.

(b) When the size of a figure is given, it is said to be given in magnitude, such as a square whose side is of given length.

(c) When the place which a figure occupies is known, it is said to be given in position.

(d) Any two equilateral triangles are similar.

[1]Adapted from: https://en.wikipedia.org/wiki/Similarity_(geometry)

2. A segment is said to be cut at a point *in extreme and mean ratio* when the whole segment is to the greater segment as the greater segment is to the lesser segment. This ratio is also referred to as the *golden ratio*.

Specifically, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. The figure on the right illustrates the geometric relationship. Expressed algebraically, for quantities a and b with $a > b > 0$,

$$\frac{a+b}{a} = \frac{a}{b} \stackrel{\text{def}}{=} \varphi$$

where the Greek letter phi (φ or ϕ) represents the golden ratio.

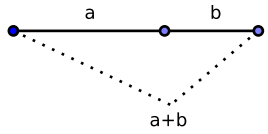


FIGURE 6.1.1. The golden ratio

Its value is:

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$$

The golden ratio also is called the *golden mean* or *golden section* (Latin: *sectio aurea*). Other names include *medial section*, *divine proportion*, *divine section* (Latin: *sectio divina*), *golden proportion*, *golden cut*, and *golden number*.[2]

3. If three quantities of the same kind are in continued proportion, the middle term is called a mean proportional between the other two. Numbers in continued proportion are also said to be in geometrical progression.

4. If four quantities of the same kind are in continued proportion, the two middle terms are called two mean proportionals between the other two.

5. The altitude of any figure is the length of the perpendicular from its highest point to its base.

6. Two corresponding angles of two figures have the sides about them *reciprocally proportional* when a side of the first is to a side of the second as the remaining side of the second is to the remaining side of the first.

[2]Much of this section comes from `https://en.wikipedia.org/wiki/Golden_ratio`.

This is equivalent to saying that a side of the first is to a side of the second in the reciprocal ratio of the remaining side of the first to the remaining side of the second.

7. *Similar figures* are said to be similarly constructed upon given segments when these segments are corresponding sides of the figures.

8. *Corresponding points* in the planes of two similar figures are such that segments constructed from them to the vertices of the two figures are proportional to the corresponding sides of the two figures. See Fig. 6.1.2.

9. The point O in Fig. 6.1.2 is called the center of similitude of the figures. It is also called their double point.

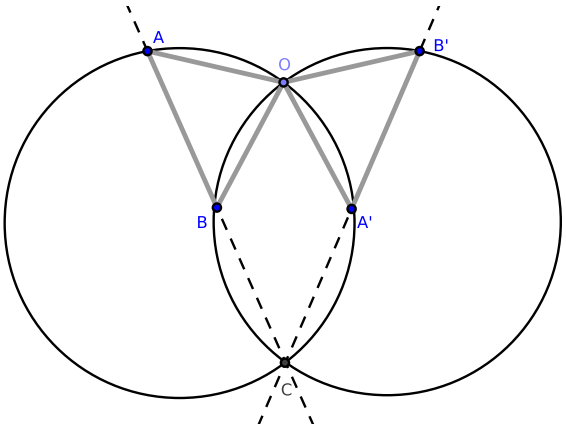


FIGURE 6.1.2. [Def 6.9] See also [6.20, #2]

6.2. Propositions from Book VI

PROPOSITION 6.1. *PROPORTIONAL TRIANGLES AND PARALLELOGRAMS.*

The areas of triangles and parallelograms which have the same altitude are proportional to their bases.

PROOF. Construct $\triangle ACB$ and $\triangle ACD$; also construct $\boxdot EACB$ and $\boxdot AFDC$ each with the same height as the previously constructed triangles. We claim that

$$\overline{BC} : \overline{CD} = \triangle ACB : \triangle ACD = \boxdot EACB : \boxdot AFDC$$

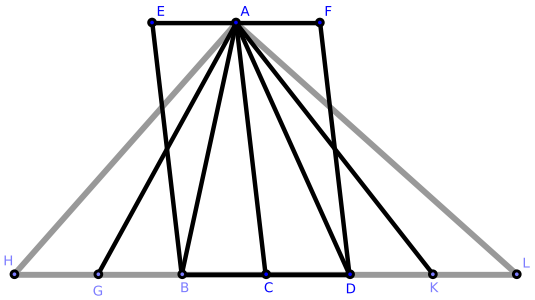


FIGURE 6.2.1. [6.1]

Extend $\overline{BD}$ in both directions to the H and L. Construct any finite number of segments, each equal in the length to $\overline{BC}$: in this proof, we construct $\overline{BG}$ and $\overline{GH}$. Similarly, construct an equal number of segments (in this proof, $\overline{DK}$ and $\overline{KL}$) where each are equal in the length to $\overline{CD}$. Also construct $\overline{AG}$, $\overline{AH}$, $\overline{AK}$, and $\overline{AL}$.

Since $\overline{BC} = \overline{BG} = \overline{GH}$, by [1.38] $\triangle ACB = \triangle ABG = \triangle AGH$. Similarly, if $\overline{CH} = k \cdot \overline{CB}$ (where $k > 1$), we also have that $\triangle ACH = k \cdot \triangle ACB$. And again, if $\overline{CL} = m \cdot \overline{CD}$ (where $m > 1$), we also have that $\triangle ACL = m \cdot \triangle ACD$.

Finally, we also have that $\overline{CH} = n \cdot \overline{CL}$ (where $n > 0$) which implies that $\triangle ACH = n \cdot \triangle ACL$. Hence,

$$\begin{array}{rcll} \overline{CH} : \overline{CL} & = & \triangle ACH : \triangle ACL & \Rightarrow \\ k \cdot \overline{BC} : m \cdot \overline{CD} & = & (k \cdot \triangle ACB) : (m \cdot \triangle ACD) & \Rightarrow \\ \overline{BC} : \overline{CD} & = & \triangle ACB : \triangle ACD & \end{array}$$

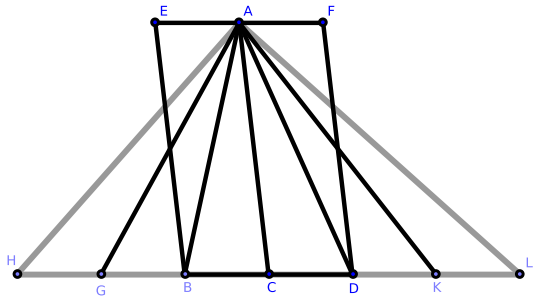


FIGURE 6.2.2. [6.1]

By [1.41], $\boxdot EBCA = 2 \cdot \triangle ACB$ and $\boxdot ACDF = 2 \cdot \triangle ACD$. Therefore

$$\begin{aligned} 2 \cdot \overline{BC} : 2 \cdot \overline{CD} &= \boxdot EBCA : \boxdot AFCD \quad \Rightarrow \\ \overline{BC} : \overline{CD} &= \boxdot EBCA : \boxdot AFCD \end{aligned}$$

This proves our claim. □

PROPOSITION 6.2. *PROPORTIONALITY OF SIDES OF TRIANGLES.*

A segment within a triangle is constructed parallel to one of the sides of a triangle if and only if the segment divides the sides of the triangle proportionally.

PROOF. Construct $\triangle ABC$ and $\overline{DE}$ where D is on $\overline{AB}$ and E is on $\overline{AC}$. We claim that $\overline{DE} \parallel \overline{BC}$ if and only if $\overline{AD} : \overline{DB} = \overline{AE} : \overline{EC}$.

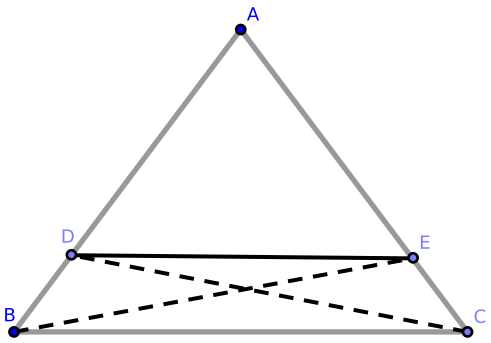


FIGURE 6.2.3. [6.2]

First, suppose that $\overline{DE} \parallel \overline{BC}$ in $\triangle ABC$. We claim that $\overline{AD} : \overline{DB} = \overline{AE} : \overline{EC}$.

Construct $\overline{BE}$ and $\overline{CD}$, and consider $\triangle BDE$ and $\triangle CED$: each shares base $\overline{DE}$ and stands between the parallels $\overline{BC}$ and $\overline{DE}$. By [1.37], $\triangle BDE = \triangle CDE$. By [5.7], $\triangle ADE : \triangle BDE = \triangle ADE : \triangle CDE$.

By [6.1], $\triangle ADE : \triangle BDE = \overline{AD} : \overline{DB}$ and $\triangle ADE : \triangle CDE = \overline{AE} : \overline{EC}$. Since $\triangle BDE = \triangle CDE$, it follows that $\overline{AD} : \overline{DB} = \overline{AE} : \overline{EC}$.

Now suppose that $\overline{AD} : \overline{DB} = \overline{AE} : \overline{EC}$. We claim $\overline{DE} \parallel \overline{BC}$.

By [6.1], $\overline{AD} : \overline{DB} = \triangle ADE : \triangle BDE$ and $\overline{AE} : \overline{EC} = \triangle ADE : \triangle CDE$. Since $\overline{AD} : \overline{DB} = \overline{AE} : \overline{EC}$ by hypothesis, $\triangle ADE : \triangle BDE = \triangle ADE : \triangle CDE$.

By [5.9], $\triangle BDE = \triangle CDE$. These triangles also stand on the same base $\overline{DE}$ as well as on the same side of $\overline{DE}$. By [1.39], they stand between the same parallels, and so $\overline{DE} \parallel \overline{BC}$. □

Exercise.

1. If two segments are cut by three or more parallels, the intercepts on one are proportional to the corresponding intercepts on the other.

PROPOSITION 6.3. *ANGLES AND PROPORTIONALITY OF TRIANGLES.*

A line bisects an angle of a triangle if and only if the line divides the side opposite the angle into segments proportional to the adjacent sides.

PROOF. Construct $\triangle ABC$ and $\overleftrightarrow{AD}$ where D is a point on $\overline{AC}$. We claim that $\overline{AD}$ bisects $\angle BAC$ if and only if $\overline{BD} : \overline{DC} = \overline{BA} : \overline{AC}$.

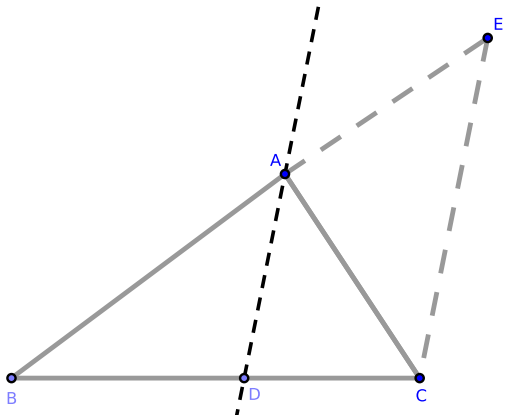


FIGURE 6.2.4. [6.3]

Suppose that $\overline{AD}$ bisects $\angle BAC$ of a triangle $\triangle ABC$. We claim that $\overline{BD} : \overline{DC} = \overline{BA} : \overline{AC}$.

Construct $\overline{CE}$ such that $\overline{CE} \parallel \overline{AD}$. Extend $\overline{BA}$ to intersect $\overline{CE}$ at E. Because $\overline{BE}$ intersects the parallels $\overline{AD}$ and $\overline{EC}$, we have that $\angle BAD = \angle AEC$ [1.29].

Because $\overline{AC}$ intersects the parallels $\overline{AD}$ and $\overline{EC}$, we have that $\angle ACE = \angle DAC$. By hypothesis, $\angle DAC = \angle BAD$. Therefore,

$$\angle ACE = \angle DAC = \angle BAD = \angle AEC$$

Consider $\triangle ACE$: by [1.6], $\overline{AE} = \overline{AC}$.

Again, because $\overline{AD} \parallel \overline{EC}$, where EC is one of the sides of the triangle $\triangle BEC$, we have that $\overline{BD} : \overline{DC} = \overline{BA} : \overline{AE}$ [6.2]. Since $\overline{AE} = \overline{AC}$ by the above, it follows that $\overline{BD} : \overline{DC} = \overline{BA} : \overline{AC}$.

Now suppose that $\overline{BD} : \overline{DC} = \overline{BA} : \overline{AC}$. We wish to prove that $\angle BAC$ is bisected by $\overline{AD}$.

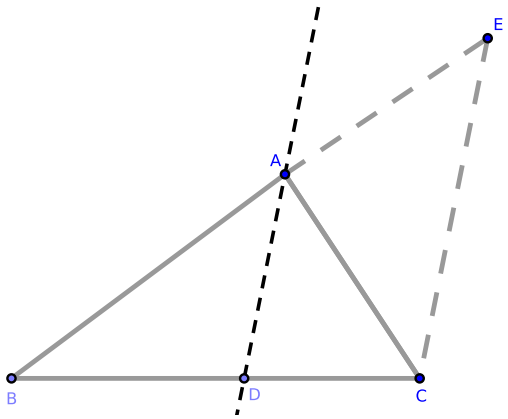


FIGURE 6.2.5. [6.3]

Let the same construction be made as above. Because $\overline{AD} \parallel \overline{EC}$, by [6.2] $\overline{BA} : \overline{AE} = \overline{BD} : \overline{DC}$.

But $\overline{BD} : \overline{DC} = \overline{BA} : \overline{AC}$ by hypothesis. By [5.11], it follows that $\overline{BA} : \overline{AE} = \overline{BA} : \overline{AC}$, and so $\overline{AE} = \overline{AC}$ by [5.9].

Consider $\triangle ACE$: by [Cor. 1.6.1], $\angle AEC = \angle ACE$. By [1.29], we also have that $\angle AEC = \angle BAD$ and that $\angle ACE = \angle DAC$. Thus $\angle BAD = \angle DAC$.

Since $\angle BAC = \angle BAD + \angle DAC$, it follows that $\angle BAC$ is bisected by $\overline{AD}$. □

Exercises.

1. If the segment $\overline{AD}$ bisects the external vertical angle $\angle CAE$, then $\overline{BA} : \overline{AC} = \overline{BD} : \overline{DC}$, and conversely.

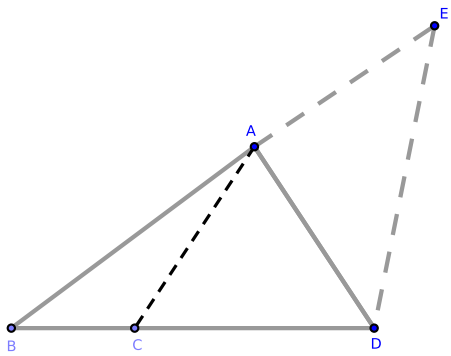


FIGURE 6.2.6. [6.3], #1

Construct $\overline{AE} = \overline{AC}$. Also construct $\overline{ED}$. Then the triangles $\triangle ACD$, $\triangle AED$ are evidently congruent; therefore the angle $\angle EDB$ is bisected, and hence $\overline{BA} : \overline{AE} = \overline{BD} : \overline{DE}$ and $\overline{BA} : \overline{AC} = \overline{BD} : \overline{DC}$ [6.3].

2. Prove #1 without using [6.3], and then prove [6.3] using #1.

3. The internal and the external bisectors of the vertical angle of a triangle divide the base harmonically.

4. Any segment intersecting the legs of any angle is cut harmonically by the internal and external bisectors of the angle.

5. Any segment intersecting the legs of a right angle is cut harmonically by any two lines through its vertex which make equal angles with either of its sides.

6. If the base of a triangle is given in number and position and if the ratio of the sides is also given, then the locus of the vertex is a circle which divides the base harmonically in the ratio of the sides.

7. If a, b, c denote the sides of a triangle $\triangle ABC$, and D, D' are the points where the internal and external bisectors of A meet BC, then prove that $DD' = \frac{2abc}{b^2-c^2}$.

8. In the same case as #7, if E, E', F, F' are points similarly determined on the sides CA, AB, respectively, prove that

$$\begin{aligned} \frac{1}{DD'} + \frac{1}{EE'} + \frac{1}{FF'} &= 0 \\ \frac{a^2}{DD'} + \frac{b^2}{EE'} + \frac{c^2}{FF'} &= 0 \end{aligned}$$

PROPOSITION 6.4. *EQUIANGULAR TRIANGLES I.*

In equiangular triangles, the sides about the equal angles are proportional where the corresponding sides stand opposite the equal angles.

PROOF. Let $\triangle ABC$ and $\triangle DCE$ be equiangular triangles where $\angle ABC = \angle DCE$, $\angle BAC = \angle CDE$, and $\angle ACB = \angle CED$. We claim that

$$\overline{BA} : \overline{AC} \quad = \quad \overline{CD} : \overline{DE}$$

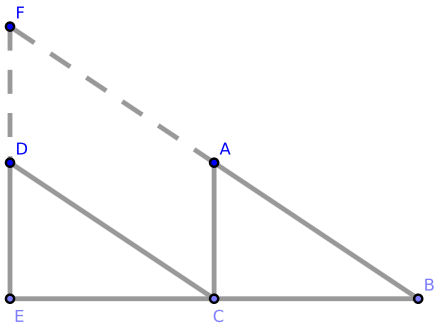


FIGURE 6.2.7. [6.4]

Place $\overline{BC}$ such that $\overline{BE} = \overline{BC} \oplus \overline{CE}$. Since $\angle ABC + \angle ACB$ is less than two right angles [1.17] and $\angle ACB = \angle DEC$, it follows that $\angle ABC + \angle DEC$ is less than two right angles. Hence when $\overline{BA}$ and $\overline{DE}$ are extended, they will intersect at F.

By [1.28], since $\angle DCE = \angle ABC$, $\overline{DC} \parallel \overline{FB}$; and since $\angle DEC = \angle ACB$, $\overline{AC} \parallel \overline{FE}$. Therefore $\boxdot FACD$ is a parallelogram; by [1.34], $\overline{FA} = \overline{DC}$ and $\overline{AC} = \overline{FD}$.

Consider $\triangle FBE$: since $\overline{AC} \parallel \overline{FE}$, by [6.2] $\overline{BA} : \overline{AF} = \overline{BC} : \overline{CE}$. But $\overline{AF} = \overline{CD}$, and so $\overline{BA} : \overline{CD} = \overline{BC} : \overline{CE}$. By [5.16], $\overline{BA} : \overline{BC} = \overline{CD} : \overline{CE}$.

Similarly, since $\overline{CD} \parallel \overline{BF}$, by [6.2] $\overline{BC} : \overline{CE} = \overline{FD} : \overline{DE}$. But $\overline{FD} = \overline{AC}$, and so $\overline{BC} : \overline{CE} = \overline{AC} : \overline{DE}$; or $\overline{BC} : \overline{AC} = \overline{CE} : \overline{DE}$.

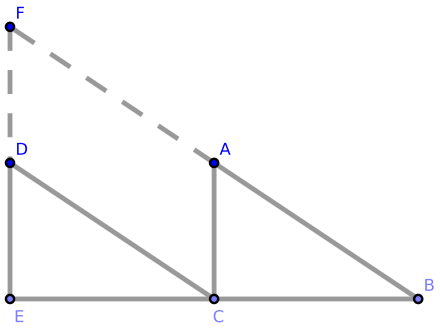
Since

$$\begin{aligned} \overline{BA} : \overline{BC} &= \overline{CD} : \overline{CE} \\ \overline{BC} : \overline{AC} &= \overline{CE} : \overline{DE} \end{aligned}$$

by [5.22], it follows that

$$\overline{BA} : \overline{AC} = \overline{CD} : \overline{DE}$$

This proves our claim. □



REMARK. It can be shown by this proposition that two lines which meet at infinity are parallel. Let I denote the point at infinity through which the two given lines pass, and construct any two parallels intersecting them in the points A, B and A', B'. Then the triangles $\triangle AIB$, $\triangle A'IB'$ are equiangular. Therefore, $\overline{AI} : \overline{AB} = \overline{A'I} : \overline{A'B'}$ where the first term of the proportion is equal to the third. By [5.14], the second term $\overline{AB}$ is equal to the fourth $\overline{A'B'}$, and, being parallel to it, the lines $\overline{AA'}$, $\overline{BB'}$ are parallel by [1.43].

Exercises.

1. If two circles intercept equal chords $\overline{AB}$, $\overline{A'B'}$ on any secant, the tangents $\overleftrightarrow{AT}$, $\overleftrightarrow{A'T}$ to the circles at the points of intersection are to one another as the radii of the circles.

2. If two circles intercept on any secant chords that have a given ratio, the tangents to the circles at the points of intersection have a given ratio, namely, the ratio compounded of the direct ratio of the radii and the inverse ratio of the chords.

3. Being given a circle and a line, prove that a point may be found such that the rectangle of the perpendiculars falling on the line from the points of intersection of the circle with any chord through the point shall be given.

4. If $\overline{AB}$ is the diameter of a semicircle ADB and $\overline{CD} \perp \overline{AB}$, construct through A a chord $\overline{AF}$ of the semicircle meeting $\overline{CD}$ at E such that the ratio $\overline{CE} : \overline{EF}$ may be given.

PROPOSITION 6.5. *EQUIANGULAR TRIANGLES II.*

If two triangles have proportional sides, then the triangles are equiangular with the equal angles opposite the corresponding sides.

PROOF. Construct $\triangle ABC$ and $\triangle DEF$ such that $\overline{BA} : \overline{AC} = \overline{ED} : \overline{DF}$ and $\overline{AC} : \overline{CB} = \overline{DF} : \overline{FE}$. We claim that:

(1)$\triangle ABC$ and $\triangle DEF$ are equiangular;

(2) equal angles stand opposite corresponding sides.

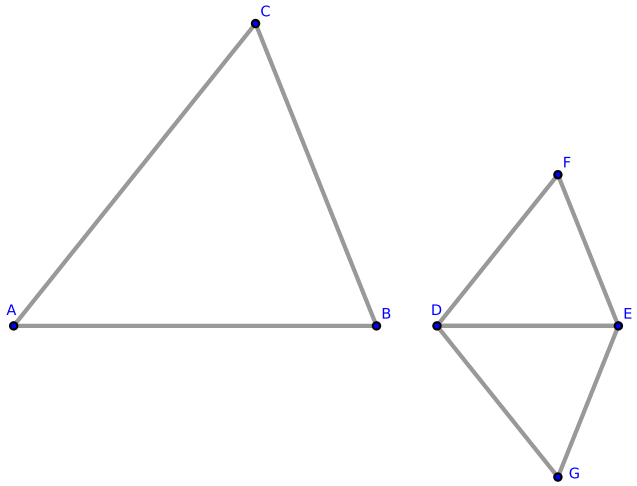


FIGURE 6.2.8. [6.5]

At D and E, construct the angles $\angle EDG = \angle BAC$ and $\angle DEG = \angle ABC$. By [1.32], $\triangle ABC$ and $\triangle DEG$ are equiangular. By [6.4], $\overline{BA} : \overline{AC} = \overline{ED} : \overline{DG}$.

Since $\overline{BA} : \overline{AC} = \overline{ED} : \overline{DF}$ by hypothesis, it follows that $\overline{DG} = \overline{DF}$. Similarly, $\overline{EG} = \overline{EF}$.

Consider $\triangle EDF$ and $\triangle EDG$: each shares side $\overline{ED}$, $\overline{DG} = \overline{DF}$, and $\overline{EG} = \overline{EF}$. By [1.8], $\triangle EDF \cong \triangle EDG$, and so $\triangle EDF$ and $\triangle EDG$ are equiangular (hence $\angle EDG = \angle EDF$).

But $\triangle EDG$ is equiangular to $\triangle ABC$ by construction. Therefore, $\triangle EDF$ is equiangular to $\triangle ABC$ (claim 1).

Since $\angle BAC$ stands between $\overline{BA}$ and $\overline{AC}$, and $\angle EDF$ stands between $\overline{ED}$ and $\overline{DF}$, claim 2 is proven, which completes the proof. □

COROLLARY. *6.5.1. Two triangles are equiangular if and only if the sides about the equal angles are proportional where the corresponding sides stand opposite the equal angles.*

REMARK. In [Def. 6.1], two conditions are laid down as necessary for the similitude of polygons:

(a) The equality of angles;

(b) The proportionality of sides.

Now by [6.4] and [6.5], we see that if two triangles possess either condition, they also possess the other. Triangles are unique in this respect. In all other polygons, one of these conditions may exist without the other. Thus two quadrilaterals may have their sides proportional without having equal angles, or vice verse.

PROPOSITION 6.6. *EQUIANGULAR TRIANGLES III.*

If two triangles have one angle in one triangle equal to one angle in the other triangle and the sides about these angles are proportional, then the triangles are equiangular and have those angles equal which stand opposite to their corresponding sides.

PROOF. Construct $\triangle ABC$ and $\triangle DEF$ such that:

(1) $\angle BAC$ in $\triangle ABC$ is equal in measure to $\angle EDF$ is $\triangle DEF$;

(2) $\overline{BA} : \overline{AC} = \overline{ED} : \overline{DF}$

We claim that $\triangle ABC$ and $\triangle DEF$ are equiangular and have those angles equal which stand opposite to their corresponding sides.

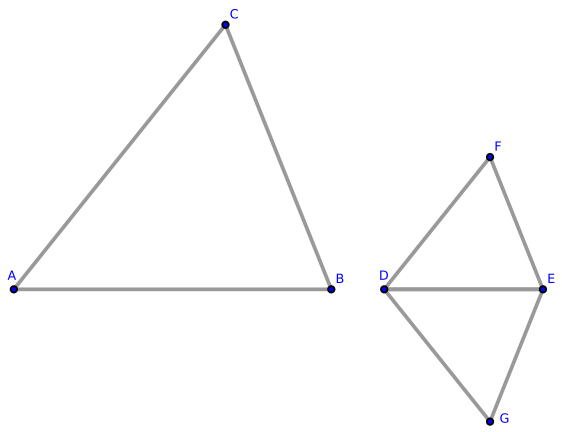


FIGURE 6.2.9. [6.6]

Recreate the construction as in [6.5]. By [6.4], $\overline{BA} : \overline{AC} = \overline{ED} : \overline{DG}$; by hypothesis, $\overline{BA} : \overline{AC} = \overline{ED} : \overline{DF}$, and so $\overline{DG} = \overline{DF}$.

Because $\angle EDG = \angle BAC$ by construction and $\angle BAC = \angle EDF$ by hypothesis, we have that $\angle EDG = \angle EDF$.

Consider $\triangle EDG$ and $\triangle EDF$: each shares side $\overline{DE}$, $\overline{DG} = \overline{DF}$, and $\angle EDG = \angle EDF$. By [1.4],$\triangle EDF \cong \triangle EDG$, and so $\triangle EDF$ and $\triangle EDG$ are equiangular.

But $\triangle EDG$ is equiangular to $\triangle BAC$ by construction, and so $\triangle EDF$ is equiangular to $\triangle BAC$.

Finally, $\triangle ABC$ and $\triangle DEF$ have equal angles which stand opposite to their corresponding sides. This proves our claim. □

REMARK. As in the case of [6.4], an immediate proof of [6.6] can be obtained from [6.2].

PROPOSITION 6.7. *SIMILAR TRIANGLES I.*

If two triangles each have one angle equal to one angle in the other, if the sides about two other angles are proportional, and if the remaining angles of the same species (i.e., either both acute or both not acute), then the triangles are similar.

PROOF. Construct $\triangle ABC$ and $\triangle DEF$ where $\angle BAC = \angle EDF$, $\overline{AB} : \overline{BC} = \overline{DE} : \overline{EF}$, and $\angle EFD$ and $\angle BCA$ are either both acute or both not acute. We claim that $\triangle ABC \sim \triangle DEF$.

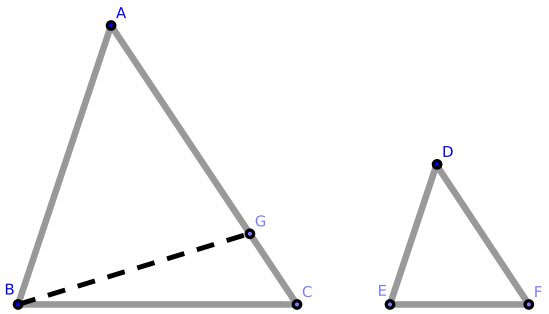


FIGURE 6.2.10. [6.7]

If $\angle ABC \neq \angle DEF$, then one angle must be greater than the other. Wlog, suppose $\angle ABC > \angle DEF$ such that $\angle ABG = \angle DEF$, from which it follows that $\angle EFD = \angle BGA$.

Consider $\triangle ABG$ and $\triangle DEF$: $\angle BAG = \angle EDF$ and $\angle ABG = \angle DEF$. By [1.32], $\triangle ABG$ and $\triangle DEF$ are equiangular. By [6.4], $\overline{AB} : \overline{BG} = \overline{DE} : \overline{EF}$.

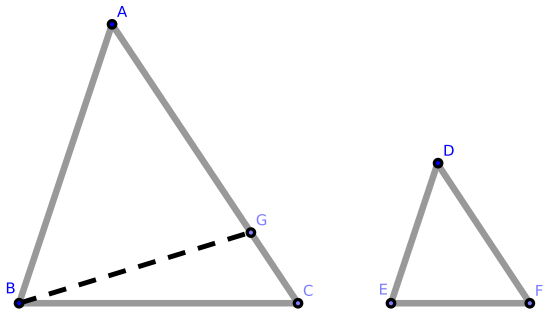
Since $\overline{AB} : \overline{BC} = \overline{DE} : \overline{EF}$ by hypothesis, $\overline{BG} = \overline{BC}$.

Consider $\triangle CBG$: by the above, the triangle is isosceles. By [1.5], $\angle BGC = \angle BCG$. And by [Cor. 1.17.1], every triangle has at least two acute angles; it follows that $\angle BGC$ and $\angle BCG$ are both acute. Since $\angle BGC$ is acute, $\angle BGA$ is obtuse.

By hypothesis, $\angle EFD = \angle BGA$, and so $\angle EFD$ is obtuse; since $\angle BCG = \angle BCA$, $\angle BCA$ is acute. However, $\angle EFD$ and $\angle BCA$ are either both acute or both obtuse by hypothesis, a contradiction.

Thus $\angle ABC = \angle DEF$. Since $\angle BAC = \angle EDF$ by hypothesis, by [1.32] $\triangle ABC$ and $\triangle EFD$ are equiangular.

Since $\overline{AB} : \overline{BC} = \overline{DE} : \overline{EF}$ by hypothesis, $\triangle ABC \sim \triangle DEF$. This proves our claim. □



REMARK. Wlog, we could also construct $\triangle ABC$ and $\triangle DEF$ where $\angle BAC = \angle EDF$, $\overline{BC} : \overline{CA} = \overline{EF} : \overline{FD}$, and $\angle ABC$ and $\angle DEF$ are either both acute or both not acute, among other options.

COROLLARY. *6.7.1. If $\triangle ABC$ and $\triangle DEF$ each have two sides proportional to two sides in the other triangle, then $\overline{AB} : \overline{BC} = \overline{DE} : \overline{EF}$, the angles at points A and D opposite one pair of corresponding sides are equal, and the angles at points C and F opposite the other are either equal or supplemental. This proposition is nearly identical with [6.7].*

COROLLARY. *6.7.2. If either of the angles at points C and F are right, the other angle must be right.*

Exercises.

1. Prove [Cor. 6.7.1].

2. Prove [Cor. 6.7.2].

3. Prove the Transitivity of Similar Triangles, i.e., if $\triangle ABC \sim \triangle DEF$ and $\triangle DEF \sim \triangle GHI$, then $\triangle ABC \sim \triangle GHI$.

PROPOSITION 6.8. *SIMILARITY OF RIGHT TRIANGLES.*

The triangles formed by dividing a right triangle by the perpendicular from the right angle to the hypotenuse are similar to the first triangle and to each other.

PROOF. Construct right triangle $\triangle ACB$, and construct $\overline{CD}$ such that D is on $\overline{AB}$ and $\overline{CD} \perp \overline{AB}$. Consider $\triangle ACD$ and $\triangle BCD$: we claim that $\triangle ACD \sim \triangle ACB$, $\triangle ACB \sim \triangle BCD$, and $\triangle ACD \sim \triangle BCD$.

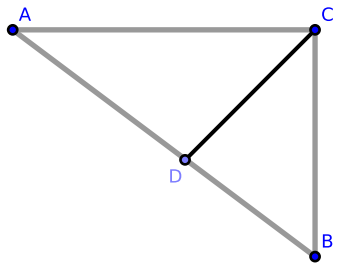


FIGURE 6.2.11. [6.8]

Consider $\triangle ACD$ and $\triangle ACB$: the triangles share $\angle BAC$, and $\angle ADC = \angle ACB$ since each are right. Hence, the triangles are equiangular by [1.32].

By [6.4], $\triangle ACD \sim \triangle ACB$. $\triangle ACB \sim \triangle BCD$ can by shown *mutatis mutandis*. By [Lemma 6.7.3], $\triangle ACD \sim \triangle BCD$, which proves our claim. □

COROLLARY. *6.8.1. The perpendicular $\overline{CD}$ is a mean proportional between the segments $\overline{AD}$ and $\overline{DB}$ of the hypotenuse. Since $\triangle ADC$ and $\triangle CDB$ are equiangular, $\overline{AD} : \overline{DC} = \overline{DC} : \overline{DB}$. Thus $\overline{DC}$ is a mean proportional between $\overline{AD}$ and$\overline{DB}$ (Def. 6.3).*

COROLLARY. *6.8.2. $\overline{BC}$ is a mean proportional between $\overline{AB}$ and $\overline{BD}$; also, $\overline{AC}$ is a mean proportional between $\overline{AB}$ and $\overline{AD}$.*

COROLLARY. *6.8.3. The segments $\overline{AD}$ and $\overline{DB}$ are in the duplicate of $\overline{AC} : \overline{CB}$; in other words, $\overline{AD} : \overline{DB} = (\overline{AC})^2 : (\overline{CB})^2$.*

COROLLARY. *6.8.4.* $\overline{BA} : \overline{AD}$ *are in the duplicate ratios of* $\overline{BA} : \overline{AC}$*, and* $\overline{AB} : \overline{BD}$ *are in the duplicate ratio of* $\overline{AB} : \overline{BC}$*. Or,* $\overline{BA} : \overline{AD} = (\overline{BA})^2 : (\overline{AC})^2$ *and* $\overline{AB} : \overline{BD} = (\overline{AB})^2 : (\overline{BC})^2$.

Exercises.

1. Prove [Cor. 6.8.1].
2. Prove [Cor. 6.8.2].
3. Prove [Cor. 6.8.3].
4. Prove [Cor. 6.8.4].

PROPOSITION 6.9. *CUTTING OFF A SUBMULTIPLE.*

From a given segment, we may cut off any required sub-multiple.

PROOF. Construct $\overline{AB}$. We wish to cut off any required sub-multiple from $\overline{AB}$.

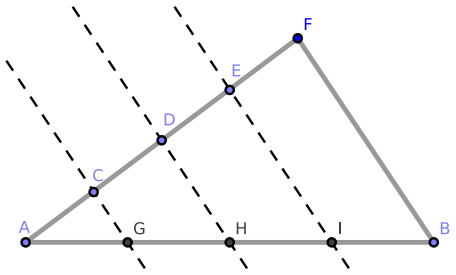


FIGURE 6.2.12. [6.9]

Suppose we wish to cut one fourth of $\overline{AB}$. Construct $\overrightarrow{AF}$ at any acute angle to $\overline{AB}$.

On $\overrightarrow{AF}$, choose point C and cut off segments $\overline{CD}$, $\overline{DE}$, and $\overline{EF}$ where

$$\overline{AC} = \overline{CD} = \overline{DE} = \overline{EF}$$

[1.3]. Construct $\overline{CG}$; also construct $\overline{DH}$, $\overline{EI}$, and $\overline{FB}$ such that each is parallel to $\overline{CG}$. We claim that $\overline{AG} = \frac{1}{4} \cdot \overline{AB}$.

Since $\overline{CG} \parallel \overline{BF}$ where $\overline{BF}$ is the side of $\triangle ABF$, by [6.2] we have that $\overline{AC} : \overline{CF} = \overline{AG} : \overline{GB}$. And so:

$$\begin{aligned}
\overline{CF} : \overline{AC} &= \overline{GB} : \overline{AG} \\
\left(\overline{AC} \oplus \overline{CF}\right) : \overline{AC} &= \left(\overline{AG} \oplus \overline{GB}\right) : \overline{AG} \\
\overline{AF} : \overline{AC} &= \overline{AB} : \overline{AG} \\
\overline{AC} : \overline{AF} &= \overline{AG} : \overline{AB}
\end{aligned}$$

But $\overline{AC} = \frac{1}{4} \cdot \overline{AF}$ by construction, and so $\overline{AG} = \frac{1}{4} \cdot \overline{AB}$. Since our choice of a fourth was arbitrary, any other required sub-multiple may similarly be cut off. □

REMARK. [1.10] is a particular case of this proposition.

Exercises.

1. Prove [6.9] using a proof by induction.

PROPOSITION 6.10. *SIMILARLY DIVIDED SEGMENTS.*

We wish to divide a given segment similarly to a given divided segment.

PROOF. Construct $\overline{AB}$ as the given segment and $\overline{AC}$ as the segment divided at points D and E where $\angle BAC$ is an acute angle. We claim that $\overline{AB}$ is divided similarly to $\overline{AC}$.

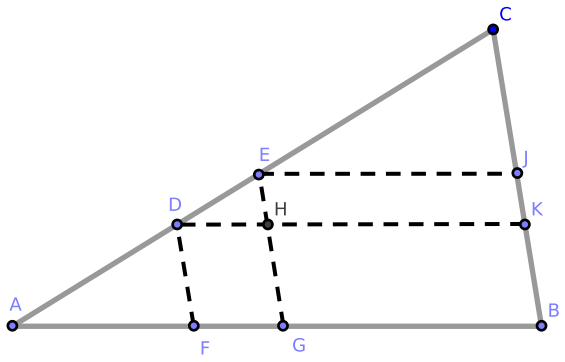


FIGURE 6.2.13. [6.10]

Construct $\overline{CB}$, $\overline{DF}$, and $\overline{EG}$ such that $\overline{DF} \parallel \overline{CB}$ and $\overline{EG} \parallel \overline{CB}$. Also construct $\overline{DK}$ where $\overline{DK}$ intersects $\overline{EG}$ at H and such that $\overline{DK} \parallel \overline{AB}$.

It follows that $\boxdot DHGF$ and $\boxdot HKBG$ are parallelograms, and so by [1.34] $\overline{DH} = \overline{FG}$ and $\overline{HK} = \overline{GB}$.

Consider $\triangle DCK$: since $\overline{EH} \parallel \overline{CK}$, by [6.2] $\overline{DE} : \overline{EC} = \overline{DH} : \overline{HK}$.

But $\overline{DH} = \overline{FG}$ and $\overline{HK} = \overline{GB}$; by [5.7], $\overline{DE} : \overline{EC} = \overline{FG} : \overline{GB}$.

Consider $\triangle AEG$: since $\overline{DF} \parallel \overline{EG}$, $\overline{AD} : \overline{DE} = \overline{AF} : \overline{FG}$.

The last two proportions give us $\overline{AD} : \overline{EC} = \overline{AF} : \overline{GB}$. This divides $\overline{AB}$ similarly to $\overline{AC}$. □

COROLLARY. *6.10.1. We may divide a given undivided segment $\overline{AB}$ similarly to a given divided segment $\overline{DE}$ by constructing $\overline{AC}$ at an acute angle to $\overline{AB}$ when $\overline{AC}$ divided into similar segments to $\overline{DE}$.*

Exercises.

1. We wish to divide a given segment $\overline{AB}$ internally or externally in the ratio of two given segments $\overline{FG}$, $\overline{HJ}$.

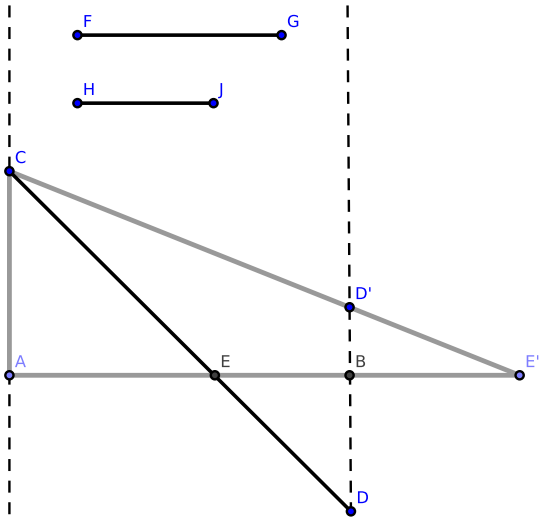


FIGURE 6.2.14. [6.11]

Through A and B construct any two parallels $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$. Construct segments $\overline{AC} = \overline{FG}$ and $\overline{BD} = \overline{HJ}$. Also construct $\overline{CD}$: we claim that $\overline{CD}$ divides $\overline{AB}$ internally at E in the ratio of $\overline{FG} : \overline{HJ}$.

2. In #1, if $\overline{BD'}$ is constructed parallel to $\overline{AC}$, then $\overline{CD}$ will cut $\overline{AB}$ externally at E in the ratio of $\overline{FG} : \overline{HJ}$.

COROLLARY. *6.10.1. The two points E, E' divide $\overline{AB}$ harmonically.*

This problem is manifestly equivalent to the following: given the sum or difference of two lines and their ratio, we wish to find the lines.

3. Any line $\overline{AE'}$ through the midpoint B of the base $\overline{DD'}$ of $\triangle DCD'$ is cut harmonically by the sides of the triangle and a parallel to the base through the vertex.

4. Given the sum of the squares on two segments and their ratio, find the segments.

5. Given the difference of the squares on two segments and their ratio, find the segments.

6. Given the base and ratio of the sides of a triangle, construct it when any of the following data is given:

(a) the area;

(b) the difference on the squares of the sides;

(c) the sum of the squares on the sides;

(d) the vertical angle;

(e) the difference of the base angles.

PROPOSITION 6.11. *PROPORTIONAL SEGMENTS I.*

Given two segments, we wish to find a third proportional segment.

PROOF. Construct $\overline{JK}$ and $\overline{LM}$. We wish to construct a segment proportional to these segments: specifically, we wish to construct a segment $\overline{X}$ such that $\overline{JK} : \overline{LM} = \overline{LM} : \overline{X}$.

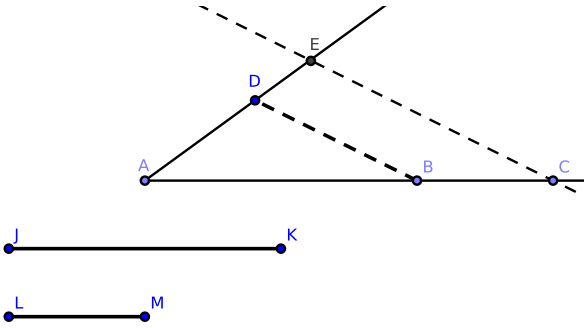


FIGURE 6.2.15. [6.11]

Construct $\overrightarrow{AC}$ and $\overrightarrow{AE}$ at an arbitrary acute angle. (The proof holds with segments of sufficient length or lines, *mutatis mutandis*.)

Cut off $\overline{AB} = \overline{JK}$, $\overline{BC} = \overline{LM}$, and $\overline{AD} = \overline{LM}$. Construct $\overline{BD}$ such that $\overline{CE} \parallel \overline{BD}$. We claim that $\overline{DE}$ is the required third proportional segment.

In $\triangle CAE$, $BD \parallel CE$; by [6.2] $\overline{AB} : \overline{BC} = \overline{AD} : \overline{DE}$. But $\overline{AB} = \overline{JK}$ and $\overline{BC} = \overline{LM} = \overline{AD}$. Hence $\overline{JK} : \overline{LM} = \overline{LM} : \overline{DE}$.

Therefore $\overline{DE}$ is a third proportional to $\overline{JK}$ and $\overline{LM}$, which proves our claim. □

REMARK. Another solution can be inferred from [6.8]. If $\overline{AD}$ and $\overline{DC}$ in that proposition are respectively equal to $\overline{JK}$ and $\overline{LM}$, then $\overline{DB}$ is the third proportional.

COROLLARY. *6.11.1 Algebraically, this problem can be written as*

$$\frac{a}{b} = \frac{b}{x} \Rightarrow x = \frac{b^2}{a}$$

where a and b are positive real numbers.

Exercises.

1. Suppose that $\triangle AO\Omega$ is a triangle where the side $\overline{A\Omega}$ is greater than $\overline{AO}$. Then if we cut $\overline{AB} = \overline{AO}$, construct $\overline{BB} \parallel \overline{AO}$, cut off $\overline{BC} = \overline{BB'}$, and so on, the series of segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, etc., are in continual proportion.

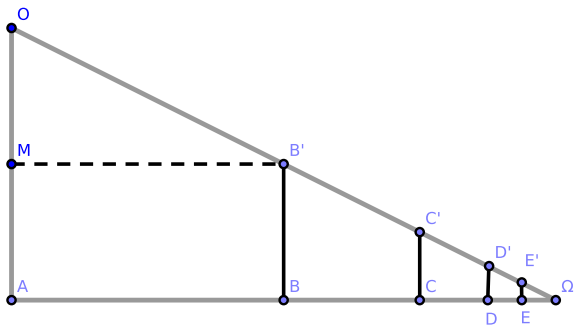


FIGURE 6.2.16. [6.2, #1]

2. Prove that $(\overline{AB} - \overline{BC}) : \overline{AB} = \overline{AB} : \overline{A\Omega}$. (Hint: This is evident by constructing $\overline{MB'} \parallel \overline{A\Omega}$.)

PROPOSITION 6.12. *PROPORTIONAL SEGMENTS II.*

We wish to find a fourth proportional to three given segments.

PROOF. Construct $\overline{AK}$, $\overline{BM}$, and $\overline{CP}$. We wish to find a fourth segment proportional to these three segments; specifically, we wish to construct a segment $\overline{X}$ such that $\overline{AK} : \overline{BM} = \overline{CP} : \overline{X}$.

Construct $\overrightarrow{DE}$, $\overrightarrow{DF}$ at an arbitrary acute angle.

Construct $\overline{DG} = \overline{AK}$, $\overline{GE} = \overline{BM}$, and $\overline{DH} = \overline{CP}$. Construct $\overline{GH}$ such that $\overline{EF} \parallel \overline{GH}$ [1.31]. We claim that $\overline{HF}$ is the required fourth proportional segment.

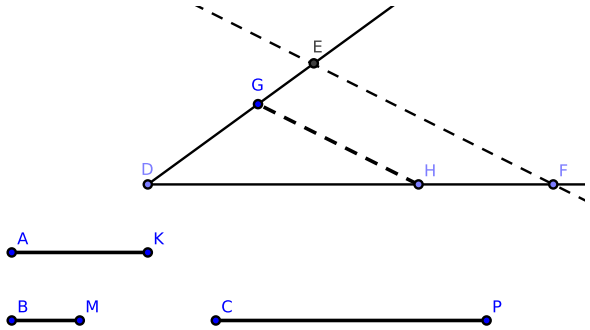


FIGURE 6.2.17. [6.12]

In $\triangle DEF$, $\overline{GH} \parallel \overline{EF}$. Therefore, $\overline{DG} : \overline{GE} = \overline{DH} : \overline{HF}$ by [6.2]. But the above equalities give us $\overline{AK} : \overline{BM} = \overline{CP} : \overline{HF}$, and so $\overline{HF}$ is a fourth proportional to $\overline{AK}$, $\overline{BM}$, and $\overline{CP}$. This proves our claim. □

COROLLARY. *6.12.1 Algebraically, this problem can be written as*

$$\frac{a}{b} = \frac{c}{x} \Rightarrow x = \frac{bc}{a}$$

where a, b, and c are positive real numbers.

PROPOSITION 6.13. *PROPORTIONAL SEGMENTS III.*

We wish to find a mean proportional between two given segments.

PROOF. Construct $\overline{EF}$ and $\overline{GH}$. We wish to find a mean proportional between these segments; specifically, we wish to construct a segment $\overline{X}$ such that $\overline{EF} : \overline{X} = \overline{X} : \overline{GH}$.

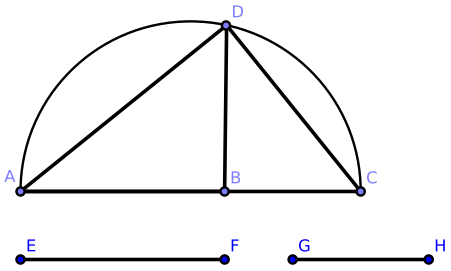


FIGURE 6.2.18. [6.13]

Construct segment $\overline{AC}$ such that $\overline{AC} = \overline{EF} + \overline{GH}$ and cut $\overline{AB} = \overline{EF}$ and $\overline{BC} = \overline{GH}$. (The proof holds on rays or lines, *mutatis mutandis*.)

On $\overline{AC}$, construct semicircle ADC. Also construct $\overline{BD} \perp \overline{AC}$ which intersects the semicircle at D. We claim that $\overline{BD}$ is the mean proportional required.

Construct $\overline{AD}$ and $\overline{DC}$. Since ADC is a semicircle, the angle $\angle ADC$ is right [3.31].

Since $\triangle ADC$ is a right triangle and $\overline{BD}$ is a perpendicular from the right angle on the hypotenuse, it follows that $\overline{BD}$ is a mean proportional between $\overline{AB}$ and $\overline{BC}$ by [6.8, Cor. 1]. That is, $\overline{BD}$ is a mean proportional between $\overline{EF}$ and $\overline{GH}$, or $\overline{EF} : \overline{BD} = \overline{BD} : \overline{GH}$.

This proves our claim. □

COROLLARY. *6.13.1. Algebraically, we have that*

$$\frac{a}{x} = \frac{x}{b} \Rightarrow \ x = \sqrt{ab}$$

where a and b are positive real numbers.

Exercises.

1. If through any point within a circle a chord is constructed which is bisected at that point, its half is a mean proportional between the segments of any other chord passing through the same point.

2. The tangent to a circle from any external point is a mean proportional between the segments of any secant passing through the same point.

3. If through the midpoint C of any arc of a circle, a secant is constructed cutting the chord of the arc at D and the circle again at E, the chord of half the arc is a mean proportional between $\overline{CD}$ and $\overline{CE}$.

4. If a circle is constructed touching another circle internally and with two parallel chords, the perpendicular from the center of the former on the diameter of the latter, which bisects the chords, is a mean proportional between the two extremes of the three segments into which the diameter is divided by the chords.

5. If a circle is constructed touching a semicircle and its diameter, the diameter of the circle is a harmonic mean between the segments into which the diameter of the semicircle is divided at the point of intersection.

6. State and prove the proposition corresponding to #4 for external contact of the circles.

PROPOSITION 6.14. *EQUIANGULAR PARALLELOGRAMS.*

We wish to prove that equiangular parallelograms are equal in area if and only if the sides about the equal angles are reciprocally proportional.

PROOF. Construct equiangular parallelograms $\boxdot HACB$ and $\boxdot CGDE$. We wish to prove that $\boxdot HACB = \boxdot CGDE$ if and only if $\overline{AC} : \overline{CE} = \overline{GC} : \overline{CB}$.

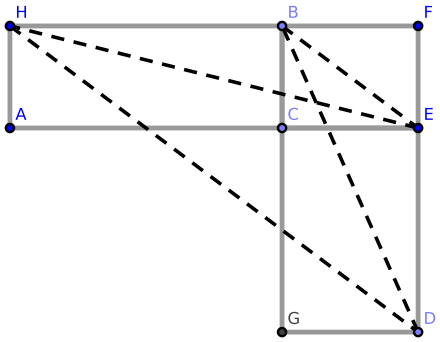


FIGURE 6.2.19. [6.14]

Suppose that $\boxdot HACB = \boxdot CGDE$; we claim that $\overline{AC} : \overline{CE} = \overline{GC} : \overline{CB}$.

Place $\boxdot HACB$ and $\boxdot CGDE$ so that $\overline{AE} = \overline{AC} \oplus \overline{CE}$ and the equal angles $\angle ACB$ and $\angle ECG$ stand vertically opposite each other.

Notice that

$$\begin{aligned} \angle ACB + \angle BCE &= \angle ECG + \angle BCE \\ \text{two right angles} &= \angle ECG + \angle BCE \end{aligned}$$

since $\angle ACB + \angle BCE$ equals two right angles by [1.13]. By [1.14], $\overline{BC} \oplus \overline{CG} = \overline{BG}$. Construct $\boxdot BCEF$.

Since $\boxdot HACB = \boxdot CGDE$,

$$\begin{aligned} \overline{AC} : \overline{CE} &= \boxdot HACB : \boxdot BCEF && [6.1] \\ \boxdot HACB : \boxdot BCEF &= \boxdot CGDE : \boxdot BCEF && \text{(hypothesis)} \\ \boxdot CGDE : \boxdot BCEF &= \overline{GC} : \overline{CB} && [6.1] \end{aligned}$$

Therefore, we have that $\overline{AC} : \overline{CE} = \overline{GC} : \overline{CB}$, which proves our first claim.

Now suppose that $\overline{AC} : \overline{CE} = \overline{GC} : \overline{CB}$. We claim that $\boxdot HACB = \boxdot CGDE$.

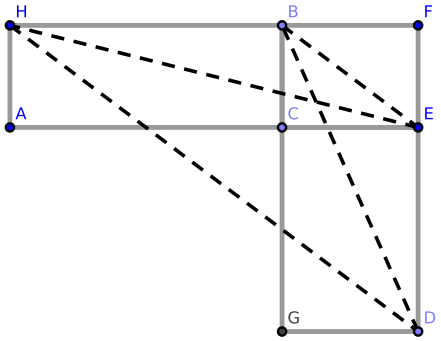


FIGURE 6.2.20. [6.14]

By [6.1], we have that

$$\begin{aligned} \overline{AC} : \overline{CE} &= \boxdot HACB : \boxdot BCEF \\ \overline{GC} : \overline{CB} &= \boxdot CGDE : \boxdot BCEF \end{aligned}$$

Since $\overline{AC} : \overline{CE} = \overline{GC} : \overline{CB}$ by hypothesis, $\boxdot HACB : \boxdot BCEF = \boxdot CGDE : \boxdot BCEF$. By [5.9], $\boxdot HACB = \boxdot CGDE$, which proves our last claim. □

Alternatively:

PROOF. Suppose that $\boxdot HACB = \boxdot CGDE$; we claim that $\overline{AC} : \overline{CE} = \overline{GC} : \overline{CB}$.

Construct $\overline{HE}$, $\overline{BE}$, $\overline{HD}$, and $\overline{BD}$. The area of the parallelogram $\boxdot HACB = 2 \cdot \triangle HBE$, and the area of the parallelogram $\boxdot CGDE = 2 \cdot \triangle BDE$. Therefore $\triangle HBE = \triangle BDE$, and by [1.39.], $\overline{HD} \parallel \overline{BE}$. Hence $\overline{HB} : \overline{BF} = \overline{DE} : \overline{EF}$; that is, $\overline{AC} : \overline{CE} = \overline{GC} : \overline{CB}$.

Part two may be proved by reversing the above. This completes our claim. □

PROPOSITION 6.15. *TRIANGLES WITH EQUAL AREAS.*

Construct two triangles which share an equal angle. These triangles have equal area if and only if they have sides about the equal angle which are reciprocally proportional.

PROOF. Construct $\triangle ACB$ and $\triangle DCE$ where $\angle BCA = \angle DCE$. We claim that $\triangle ACB = \triangle DCE$ if and only if $\overline{AC} : \overline{CD} = \overline{EC} : \overline{CB}$.

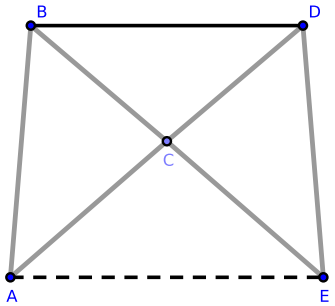


FIGURE 6.2.21. [6.15]

Suppose that $\triangle ACB = \triangle DCE$; we claim that $\overline{AC} : \overline{CD} = \overline{EC} : \overline{CB}$.

Place $\angle BCA$ and $\angle DCE$ to stand vertically opposite so that $\overline{AD} = \overline{AC} \oplus \overline{CD}$; as in the proof to [6.14], we have that $\overline{BE} = \overline{BC} \oplus \overline{CE}$. Construct $\overline{BD}$.

Since $\triangle ACB = \triangle DCE$,

$$\begin{array}{lcll} \triangle ACB : \triangle BCD & = & \triangle DCE : \triangle BCD & \\ \triangle ACB : \triangle BCD & = & \overline{AC} : \overline{CD} & [6.1] \\ \triangle DCE : \triangle BCD & = & \overline{EC} : \overline{CB} & [6.1] \end{array}$$

Therefore, $\overline{AC} : \overline{CD} = \overline{EC} : \overline{CB}$, which proves our first claim.

Now suppose that $\overline{AC} : \overline{CD} = \overline{EC} : \overline{CB}$; we claim that $\triangle ACB = \triangle DCE$.

Using the same construction, we have

$$\begin{array}{lcll} \overline{AC} : \overline{CD} & = & \overline{EC} : \overline{CB} & \text{(hypothesis)} \\ \overline{AC} : \overline{CD} & = & \triangle ACB : \triangle BCD & [6.1] \\ \overline{EC} : \overline{CB} & = & \triangle DCE : \triangle BCD & [6.1] \end{array}$$

Therefore $\triangle ACB : \triangle BCD = \triangle DCE : \triangle BCD$, and so $\triangle ACB = \triangle DCE$ [5.9]. This proves our last claim. □

REMARK. [6.15] might have been appended as a corollary to [6.14] since the triangles are the halves of equiangular parallelograms; it may also be proven by constructing $\overline{AE}$ and showing that it is parallel to $\overline{BD}$.

PROPOSITION 6.16. *PROPORTIONAL RECTANGLES.*

Four segments are proportional if and only if the rectangle contained by the extremes (i.e., the largest and the smallest segments) equals the rectangle contained by the means (i.e., the remaining segments).

PROOF. We claim that $\overline{AB} : \overline{CD} = \overline{LM} : \overline{NP}$ if and only if $\overline{AB} \cdot \overline{NP} = \overline{CD} \cdot \overline{LM}$.

Let $\overline{AB} = x$, $\overline{CD} = y$, $\overline{LM} = u$, and $\overline{NP} = v$. Then

$$\frac{x}{y} = \frac{u}{v} \Rightarrow x \cdot v = y \cdot u$$

which completes the proof. □

A geometric proof:

PROOF. Place the four segments in a concurrent position so that the extremes form one continuous segment and the means form a second continuous segment.

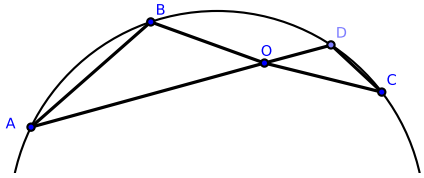


FIGURE 6.2.22. [6.16], Alternative proof

Place the four segments in the order $\overline{AO}$, $\overline{BO}$, $\overline{OD}$, and $\overline{OC}$. Construct $\overline{AB}$ and $\overline{CD}$. Because $\overline{AO} : \overline{OB} = \overline{OD} : \overline{OC}$ and $\angle AOB = \angle DOC$, the triangles $\triangle AOB$ and $\triangle COD$ are equiangular. Thus, the four points A, B, C, and D are concyclic; by [3.35], $\overline{AO} \cdot \overline{OC} = \overline{BO} \cdot \overline{OD}$. □

PROPOSITION 6.17. *LINES AND RECTANGLES.*

Three segments are proportional if and only if the rectangle contained by the endpoints is equal in area to the square of the mean.

PROOF. Construct segments $\overline{AB}$, $\overline{CD}$, and $\overline{GH}$ so that they are proportional: $\overline{AB} : \overline{CD} = \overline{CD} : \overline{GH}$. We claim that $\overline{AB} \cdot \overline{GH} = (\overline{CD})^2$.

Let $\overline{AB} = x$, $\overline{CD} = y$, and $\overline{GH} = z$. Then

$$\frac{x}{y} = \frac{y}{z} \Rightarrow \ x \cdot z = y^2$$

This proves our claim. □

REMARK. This proposition may also be inferred as a corollary to [6.16].

Exercises.

1. If a segment $\overline{CD}$ bisects the vertical angle at C of an arbitrary triangle, $\triangle ACB$, its square added to the rectangle $\overline{AD} \cdot \overline{DB}$ contained by the segments of the base is equal in area to the rectangle contained by the sides.

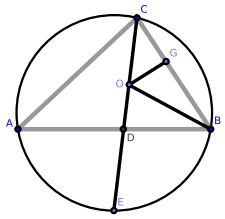


FIGURE 6.2.23. [6.17, #1]

Hint: Construct a circle about the triangle, and extend $\overline{CD}$ to intersect the circumference at E. Then show that $\triangle ACB$ $\triangle ECB$ are equiangular. Hence by [6.4] $\overline{AC} : \overline{CD} = \overline{CE} : \overline{CB}$, and therefore (by [3.35]), we have that

$$\begin{aligned} \overline{AC} \cdot \overline{CB} &= \overline{CE} \cdot \overline{CD} \\ &= (\overline{CD})^2 + \overline{CD} \cdot \overline{DE} \\ &= (\overline{CD})^2 + \overline{AD} \cdot \overline{DB} \end{aligned}$$

2. If $\overleftrightarrow{CD'}$ bisects the external vertical angle of an arbitrary triangle, $\triangle ACB$, its square subtracted from the rectangle $\overline{AD'} \cdot \overline{D'B}$ is equal in area to $\overline{AC} \cdot \overline{CB}$.

3. If a circle passing through the angle at point A of a parallelogram $\square ABCD$ intersects the two sides $\overline{AB}$, $\overline{AD}$ again at the points E, G and the diagonal AC again at F, then $\overline{AB} \cdot \overline{AE} + \overline{AD} \cdot \overline{AG} = \overline{AC} \cdot \overline{AF}$.

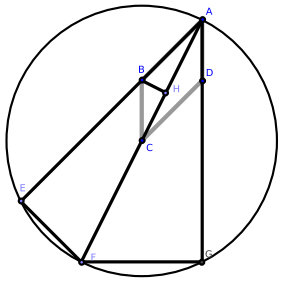


FIGURE 6.2.24. [6.17, #3]

Construct $\overline{EF}$, $\overline{FG}$, and $\angle ABH = \angle AFE$. Then the triangles $\triangle ABH$, $\triangle AFE$ are equiangular: it follows that $\overline{AB} : \overline{AH} = \overline{AF} : \overline{AE}$. Hence, $\overline{AB} \cdot \overline{AE} = \overline{AF} \cdot \overline{AH}$.

Again, it is clear that the triangles $\triangle BCH$, $\triangle GAF$ are equiangular, and therefore $\overline{BC} : \overline{CH} = \overline{AF} : \overline{AG}$, and so $\overline{BC} \cdot \overline{AG} = \overline{AF} \cdot \overline{CH}$, or $\overline{AD} \cdot \overline{AG} = \overline{AF} \cdot \overline{CH}$. But since $\overline{AB} \cdot \overline{AE} = \overline{AF} \cdot \overline{AH}$, we have that $\overline{AD} \cdot \overline{AG} + \overline{AB} \cdot \overline{AE} = \overline{AF} \cdot \overline{CH}$.

4. If $\overline{DE}$, $\overline{DF}$ are parallels to the sides of $\triangle ABC$ from any point D at the base, then $\overline{AB} \cdot \overline{AE} + \overline{AC} \cdot \overline{AF} = (\overline{AD})^2 + \overline{BD} \cdot \overline{DC}$.

5. If through a point O within a triangle $\triangle ABC$ parallels $\overline{EF}$, $\overline{GH}$, $\overline{IK}$ are constructed to the sides, the sum of the rectangles of their segments is equal to the rectangle contained by the segments of any chord of the circumscribing circle passing through O.

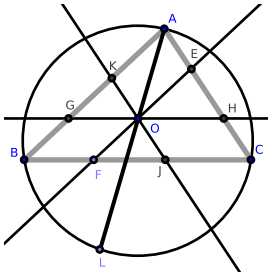


FIGURE 6.2.25. [6.17, #5]

Notice that

$$\overline{AO} \cdot \overline{AL} = \overline{AB} \cdot \overline{AK} + \overline{AC} \cdot \overline{AE}$$

and

$$(\overline{AO})^2 = \overline{AG} \cdot \overline{AK} + \overline{AH} \cdot \overline{AE} - \overline{GO} \cdot \overline{OH}$$

Hence,

$$\overline{AO} \cdot \overline{OL} = \overline{BG} \cdot \overline{AK} + \overline{CH} \cdot \overline{AE} + \overline{GO} \cdot \overline{OH}$$

or

$$\overline{AO} \cdot \overline{OL} = \overline{EO} \cdot \overline{OF} + \overline{IO} \cdot \overline{OK} + \overline{GO} \cdot \overline{OH}$$

6. The rectangle contained by the side of an inscribed square standing on the base of a triangle, and the sum of the base and altitude, is equal to twice the area of the triangle.

7. The rectangle contained by the side of an escribed square standing on the base of a triangle, and the difference between the base and altitude, is equal to twice the area of the triangle.

8. If from any point P in the circumference of a circle a perpendicular be drawn to any chord, its square is equal to the rectangle contained by the perpendiculars from the extremities of the chord on the tangent at P.

9. If O is the point of intersection of the diagonals of a cyclic quadrilateral $ABCD$, then the four rectangles $\overline{AB} \cdot \overline{BC}$, $\overline{BD} \cdot \overline{CD}$, $\overline{CD} \cdot \overline{DA}$, $\overline{DA} \cdot \overline{AB}$ are proportional to the four segments $\overline{BO}$, $\overline{CO}$, $\overline{DO}$, $\overline{AO}$.

10. ***PTOLEMY'S THEOREM.*** The sum of the rectangles of the opposite sides of a cyclic quadrilateral $ABCD$ is equal in area to the rectangle contained by its diagonals.

Construct $\angle DAO = \angle CAB$. Then $\triangle DAO$, $\triangle CAB$ are equiangular; therefore $\overline{AD} : \overline{DO} = \overline{AC} : \overline{CB}$ and so $\overline{AD} \cdot \overline{BC} = \overline{AC} \cdot \overline{DO}$. Again, the triangles $\triangle DAC$, $\triangle OAB$ are equiangular, and $\overline{CD} : \overline{AC} = \overline{BO} : \overline{AB}$ and so $\overline{AC} \cdot \overline{CD} = \overline{AC} \cdot BO$.

Hence $\overline{AD} \cdot \overline{BC} + \overline{AB} \cdot \overline{CD} = \overline{AC} \cdot \overline{BD}$.

11. If the quadrilateral $ABCD$ is not cyclic, prove that the three rectangles $AB \cdot CD$, $BC \cdot AD$, $AC \cdot BD$ are proportional to the three sides of a triangle which has an angle equal to the sum of a pair of opposite angles of the quadrilateral.

12. Prove by using [6.11] that if perpendiculars fall on the sides and diagonals of a cyclic quadrilateral from any point in the circumference of the circumscribed circle that the rectangle contained by the perpendiculars on the diagonals is equal in area to the rectangle contained by the perpendiculars on either pair of opposite sides.

13. If $\overline{AB}$ is the diameter of a semicircle, and $\overline{PA}$, $\overline{PB}$ are chords from any point P in the circumference, and if a perpendicular to $\overline{AB}$ from any point C intersects $\overline{PA}$, $\overline{PB}$ at D and E, and the semicircle at F, then $\overline{CF}$ is a mean proportional between $\overline{CD}$ and $\overline{CE}$.

PROPOSITION 6.18. *CONSTRUCTION OF A SIMILAR POLYGON.*

We wish to construct a rectilinear figure similar and similarly placed to a given rectilinear figure on a given segment.

PROOF. Construct polygon $CDEFG$ and segment AB. We wish to construct a polygon on AB which is similar to polygon $CDEFG$ and which is similarly placed.

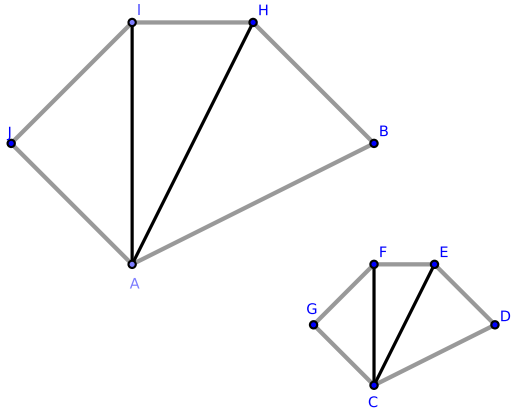


FIGURE 6.2.26. [6.18]

Construct $\overline{CE}$ and $\overline{CF}$. Also construct $\triangle ABH$ on $\overline{AB}$ such that $\triangle ABH$ is equiangular to $\triangle CDE$ and is similarly placed in regards to $\overline{CD}$; that is, construct $\angle ABH = \angle CDE$ and $\angle BAH = \angle DCE$.

Also construct $\triangle HAI$ equiangular to $\triangle ECF$ and similarly placed. Finally, construct $\triangle IAJ$ equiangular and similarly placed with $\triangle FCG$. We claim that $ABHIJ$ is the required polygon.

By construction, it is evident that the figures are equiangular, and it is only required to prove that the sides about the equal angles are proportional.

Because $\triangle ABH$ is equiangular to $\triangle CDE$, we have that $\overline{AB} : \overline{BH} = \overline{CD} : \overline{DE}$ [6.4]. Hence the sides about the equal angles at points B and D are proportional.

Again from the same triangles, we have $\overline{BH} : \overline{HA} = \overline{DE} : \overline{EC}$, and from the triangles $\triangle IHA$, $\triangle FEC$, we have $\overline{HA} : \overline{HI} = \overline{EC} : \overline{EF}$. Therefore, $\overline{BH} : \overline{HI} = \overline{DE} : \overline{EF}$, or the sides about the equal angles $\angle BHI$, $\angle DEF$ are proportional.

This result follows about the other equal angles, *mutatis mutandis*. By [Def. 6.1] and our placement of each triangle, our claim is proven. □

REMARK. In the foregoing construction, the segment $\overline{AB}$ corresponds to $\overline{CD}$, and it is evident that we may take $\overline{AB}$ to correspond to any other side of the given figure $CDEFG$.

Again, in each case, if the figure $ABHIJ$ is turned round the segment $\overline{AB}$ until it falls on the other side, it will still be similar to the figure $CDEFG$. Hence on a given segment $\overline{AB}$, there can be constructed two figures each similar to a given figure $CDEFG$ and having the given segment $\overline{AB}$ correspond to any given side $\overline{CD}$ of the given figure.

The first of the figures thus constructed is said to be directly similar, and the second is said to be inversely similar to the given figure.

COROLLARY. *6.18.1. Twice as many polygons may be constructed on $\overline{AB}$ similar to a given polygon $CDEFG$ as that figure has sides.*

COROLLARY. *6.18.2. If the figure $ABHIJ$ is applied to $CDEFG$ so that the point A coincides with C and that the segment $\overline{AB}$ is placed along $\overline{CD}$, then the points H, I, J will be respectively on the segments $\overline{CE}$, $\overline{CF}$, $\overline{CG}$. Also, the sides $\overline{BH}$, $\overline{HI}$, $\overline{IJ}$ of the one polygon will be respectively parallel to their corresponding sides $\overline{DE}$, $\overline{EF}$, $\overline{FG}$ of the other.*

COROLLARY. *6.18.3. If segments constructed from any point O in the plane of a figure to all its vertices are divided in the same ratio, the lines joining the points of division will form a new figure similar to and having every side parallel to the corresponding side of the original.*

Exercises.

1. Prove [Cor. 6.18.1].
2. Prove [Cor. 6.18.2].
3. Prove [Cor. 6.18.3].

PROPOSITION 6.19. *RATIOS OF SIMILAR TRIANGLES.*

The areas of similar triangles have a ratio equal to the square of the ratio of the triangles' corresponding sides.

PROOF. Construct $\triangle ABC$ and $\triangle DEF$ such that $\triangle ABC \sim \triangle DEF$, $\angle ABC = \angle DEF$, and $\overline{AB} : \overline{BC} = \overline{DE} : \overline{EF}$. We claim that

$$\triangle ABC : \triangle DEF = \left(\overline{BC} : \overline{EF}\right)^2$$

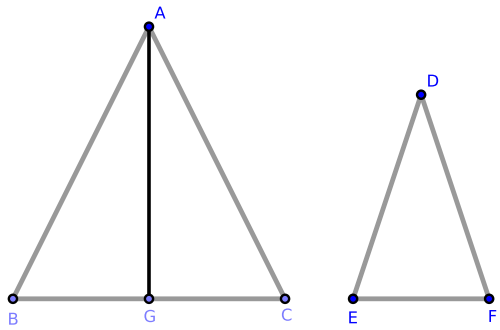


FIGURE 6.2.27. [6.19]

Construct $\overline{AG}$ such that $\overline{BC} : \overline{EF} = \overline{EF} : \overline{BG}$ ($\overline{BG}$ exists by [6.11]).

Since $\overline{AB} : \overline{BC} = \overline{DE} : \overline{EF}$ by hypothesis, $\overline{AB} : \overline{DE} = \overline{BC} : \overline{EF}$ by [5.16].

Taken together, we have the equality $\overline{AB} : \overline{DE} = \overline{EF} : \overline{BG}$ by [5.11].

Thus in $\triangle ABG$ and $\triangle DEF$, the sides about the equal angles are reciprocally proportional. By [6.15], $\triangle ABG = \triangle DEF$.

Since $\overline{BC} : \overline{EF} = \overline{EF} : \overline{BG}$, by [Def. 5.9], the duplicate ratio, we have $\overline{BC} : \overline{BG} = \left(\overline{BC} : \overline{EF}\right)^2$. Or, let $x = \overline{BC}$, $y = \overline{EF}$, and $z = \overline{BG}$. If

$$\frac{x}{y} = \frac{y}{z}$$

then

$$\frac{x}{y} \cdot \frac{y}{z} = \frac{y}{z} \cdot \frac{y}{z}$$

$$\frac{x}{z} = \left(\frac{y}{z}\right)^2$$

By [6.1], we also have $\overline{BC} \cdot \overline{BG} = \triangle ABC : \triangle ABG$, and so $\triangle ABC : \triangle ABG = \left(\overline{BC} : \overline{EF}\right)^2$.

Since $\triangle ABG = \triangle DEF$, we have $\triangle ABC : \triangle DEF = \left(\overline{BC} : \overline{EF}\right)^2$. This proves our claim. □

REMARK. [6.19] is the first of Euclid's Proposition in which the technical term "duplicate ratio" occurs. John Casey provides a similar proof which uses a new definition of the duplicate ratio of two lines: *the ratio of the squares constructed on these segments*.

PROOF. Suppose that $\triangle ABC \sim \triangle DEF$. On $\overline{AB}$ and $\overline{DE}$, construct squares ($\boxdot AGHB$ and $\boxdot DLME$, respectively), and through points C and F construct segments parallel and equal to $\overline{AB}$ and $\overline{DE}$. Extend $\overline{AG}$, $\overline{BH}$, $\overline{DL}$, and $\overline{EM}$ to points J, I, O, and N, respectively; this constructs the rectangles $\boxdot JABI$ and $\boxdot ODEN$.

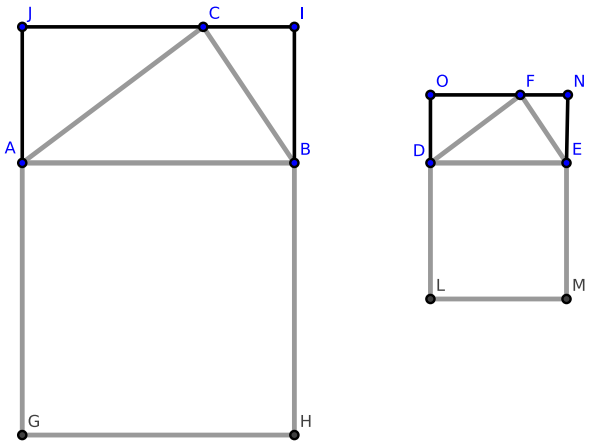


FIGURE 6.2.28. [6.19] (Casey's proof)

Clearly, $\triangle JAC$ and $\triangle ODF$ are equiangular.

By [6.4], $\overline{JA} : \overline{AC} = \overline{OD} : \overline{DF}$ and $\overline{AC} : \overline{AB} = \overline{DF} : \overline{DE}$; thus, $\overline{JA} : \overline{AB} = \overline{OD} : \overline{DE}$.

Since $\overline{AB} = \overline{AG}$ and $\overline{DE} = \overline{DL}$ by construction, $\overline{JA} : \overline{AG} = \overline{OD} : \overline{DL}$.

By [6.1], $\overline{JA} : \overline{AG} = \boxdot JABI : \boxdot AGHB$ and $\overline{OD} : \overline{DL} = \boxdot ODEN : \boxdot DLME$, and so

$$\boxdot JABI : \boxdot AGHB = \boxdot ODEN : \boxdot DLME$$

By [5.16], we have

$$\begin{aligned} \boxdot JABI : \boxdot ODEN &= \boxdot AGHB : \boxdot DLME \\ \frac{1}{2} \cdot \boxdot JABI : \frac{1}{2} \cdot \boxdot ODEN &= (\overline{AB})^2 : (\overline{DE})^2 \\ \triangle ABC : \triangle DEF &= (\overline{AB})^2 : (\overline{DE})^2 \end{aligned}$$

□

COROLLARY. *6.19.1 If three segments are proportional, then the first is to the third as the figure described on the first is to that which is similar and similarly described on the second.*[3]

Exercises.

1. If one of two similar triangles has a side that is 50% longer than the corresponding sides of the other, what is the ratio of their areas?

2. When the inscribed and circumscribed regular polygons of any common number of sides to a circle have more than four sides, the difference of their areas is less than the square of the side of the inscribed polygon.

3. Prove [Cor. 6.19.1].

[3] `http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI19.html`

PROPOSITION 6.20. *DIVISION OF SIMILAR POLYGONS.*

Similar polygons may be divided such that:

(1) they divide into the same number of similar triangles,

(2) corresponding triangles have the same ratio to one another as the polygons have to each other,

(3) the polygons have a duplicate ratio of their corresponding sides.

PROOF. Construct polygons $ABHIJ$ and $CDEFG$ such that $ABHIJ \sim CDEFG$ and sides $\overline{AB}$ and $\overline{CD}$ correspond to each other. Also construct $\overline{AH}$, $\overline{AI}$, $\overline{CE}$, and $\overline{CF}$. Our claims are stated above.

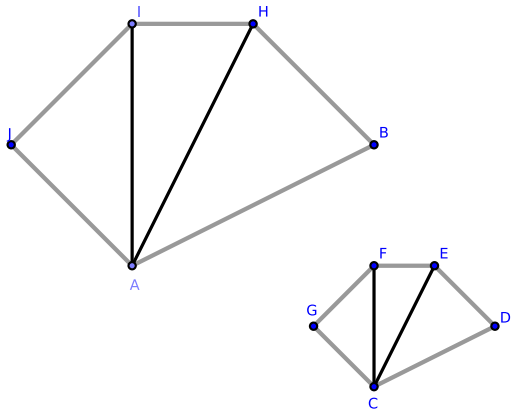


FIGURE 6.2.29. [6.20]

Claim 1: $ABHIJ$ and $CDEFG$ divide into the same number of similar triangles

Since $ABHIJ$ and $CDEFG$ are similar, by [Def. 6.1] they are equiangular and have proportional sides about their equal angles. Hence, the angle at B is equal to the angle at D, and $\overline{AB} : \overline{BH} = \overline{CD} : \overline{DE}$.

By [6.6], $\triangle ABH$ is equiangular to $\triangle CDE$, and so $\angle BHA = \angle DEC$. Since $\angle BHI = \angle DEF$ by hypothesis, $\angle AHI = \angle CEF$. Also $\overline{IH} : \overline{HB} = \overline{FE} : \overline{ED}$; since $\triangle ABH \sim \triangle CDE$, $\overline{HB} : \overline{HA} = \overline{ED} : \overline{EC}$.

It follows that $\overline{IH} : \overline{HA} = \overline{FE} : \overline{EC}$, and so $\angle IHA = \angle FEC$. Therefore, $\triangle IHA$ and $\triangle FEC$ are equiangular. Each remaining pair of corresponding triangles can also be shown to be equiangular, *mutatis mutandis*. This proves claim 1.

Claim 2: corresponding triangles within $ABHIJ$ and $CDEFG$ have the same ratio to one another as the polygons have to each other.

Since $\triangle ABH \sim \triangle CDE$, by [6.19], $\triangle ABH : \triangle CDE$ is in the duplicate ratio of $\overline{AH} : \overline{CE}$.

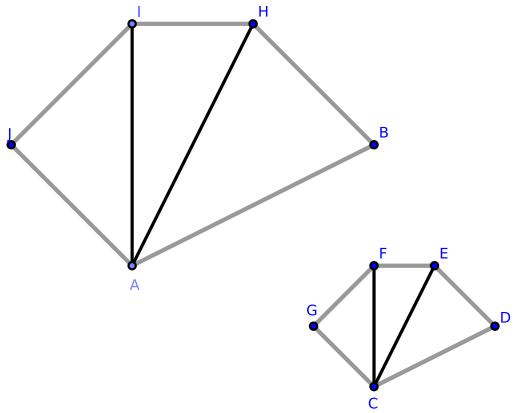


FIGURE 6.2.30. [6.20]

Also by [6.19], $\triangle AHI : \triangle CEF$ is in the duplicate ratio of $\overline{AH} : \overline{CE}$.

Hence, $\triangle ABH : \triangle CDE = \triangle AHI : \triangle CEF$. And $\triangle AHI : \triangle CEF = \triangle AIJ : \triangle CFG$, *mutatis mutandis*. It follows that

$$\frac{\triangle ABH}{\triangle CDE} = \frac{3}{3} \cdot \frac{\triangle ABH}{\triangle CDE}$$

$$\frac{\triangle ABH}{\triangle CDE} = \frac{\triangle ABH}{\triangle CDE} + \frac{\triangle AHI}{\triangle CEF} + \frac{\triangle AIJ}{\triangle CFG}$$

$$\frac{\triangle ABH}{\triangle CDE} = \frac{\text{polygon } ABHIJ}{\text{polygon } CDEFG}$$

This proves claim 2.

Claim 3: the polygons have a duplicate ratio of their corresponding sides.

By [6.19], $\triangle ABH : \triangle CDE$ is in the duplicate ratio of $\overline{AB} : \overline{CD}$. Given the above equality, we have that polygon $ABHIJ$: polygon $CDEFG$ is in the duplicate ratio of $\overline{AB} : \overline{CD}$.

The proves our claim. □

COROLLARY. *6.20.1. The perimeters of similar polygons are to one another in the ratio of their corresponding sides.*

COROLLARY. *6.20.2. As squares are to similar polygons, the duplicate ratio of two segments is equal to the ratio of the squares constructed on them.*

COROLLARY. *6.20.3. Similar portions of similar figures bear the same ratio to each other as the wholes of the figures.*

COROLLARY. *6.20.4. Similar portions of the perimeters of similar figures are to each other in the ratio of the whole perimeters.*

Homologous points in the planes of two similar figures are such, that lines drawn from them to the angular points of the two figures are proportional to the homologous sides of the two figures.

Definition: 10. *Homologous points* in the planes of two similar figures are points such that lines drawn from them to the angular points of the two figures are proportional to the homologous sides of the two figures.

Exercises.

1. Prove [Cor. 6.20.1].
2. Prove [Cor. 6.20.2].
3. Prove [Cor. 6.20.3].
4. Prove [Cor. 6.20.4].

5. If two figures are similar, prove that to each point in the plane of one there will be a corresponding point in the plane of the other.

Let $ABCD$ and $A'B'C'D'$ be the two figures and P a point inside of $ABCD$. Construct $\overline{AP}$ and $\overline{BP}$, and also construct a triangle $\triangle A'P'B'$ on $\overline{A'B'}$ similar to $\triangle APB$. Prove that segments from P' to the vertices of $A'B'C'D'$ are proportional to the lines from P to the vertices of $ABCD$.

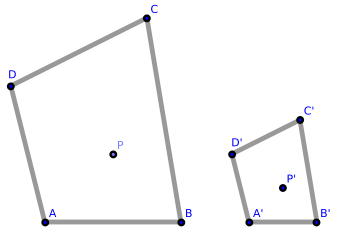


FIGURE 6.2.31. [6.20] #1

6. If two figures are similar and in the same plane, there is in the plane called a corresponding point with respect to the other (which may be regarded as belonging to either figure).

Let $\overline{AB}$, $\overline{A'B'}$ be two corresponding sides of the figures and C their point of intersection. Through the two triads of points A, A', C and B, B', C construct two circles intersecting again at the point O. Prove that O is the required point. Notice that $\triangle OAB \sim \triangle OAB$ and either may be rotated around O, so that $\overline{AB}$ and $\overline{A'B'}$ will be parallel.

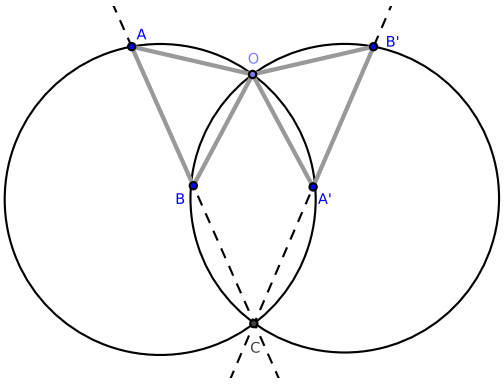


FIGURE 6.2.32. [6.20] #2

Definition: 11. The point O is called the *center of similitude of the figures*. It is also called their double point.

7. Prove that two regular polygons of n sides each have n centers of similitude.

8. If any number of similar triangles have their corresponding vertices lying on three given lines, they have a common center of similitude.

9. If two figures are directly similar and have a pair of corresponding sides parallel, every pair of corresponding sides will be parallel.

Definition: 12. Figures such as those in #9 are said to be *homothetic*. Two polygons are said to be homothetic if they are similar and their corresponding sides are parallel. If two polygons are homothetic, then the lines joining their corresponding vertices meet at a point.[4]

10. If two figures are homothetic, the segments joining corresponding vertices are concurrent, and the point of concurrence is the center of similitude of the figures.

11. If two polygons are directly similar, either may be turned round their center of similitude until they become homothetic, and this may be done in two different ways.

12. Prove that sectors of circles having equal central angles are similar figures.

13. As any two points of two circles may be regarded as corresponding, two circles have in consequence an infinite number of centers of similitude. Their locus is the circle, whose diameter is the line joining the two points for which the two circles are homothetic.

14. The areas of circles are to one another as the squares of their diameters. For they are to one another as the similar elementary triangles into which they are divided, and these are as the squares of the radii.

15. The circumferences of circles are proportional to their diameters (see [6.20, Cor. 1]).

16. The circumference of sectors having equal central angles are proportional to their radii. Hence if a, a' denote the arcs of two sectors which stand opposite equal angles at the centers, and if r, r' are their radii, then we have that $\frac{a}{r} = \frac{a'}{r'}$.

17. The area of a sector of a circle is equal to half the rectangle contained by the arc of the sector and the radius of the circle.

[4] http://americanhistory.si.edu/collections/search/object/nmah_694635

PROPOSITION 6.21. *TRANSITIVITY OF SIMILAR POLYGONS.*

Polygons which are similar to the same figure are similar to one another.

PROOF. Construct polygons ABC, DEF, and GHI such that $ABC \sim GHI$ and $DEF \sim GHI$. We claim that $ABC \sim DEF$.

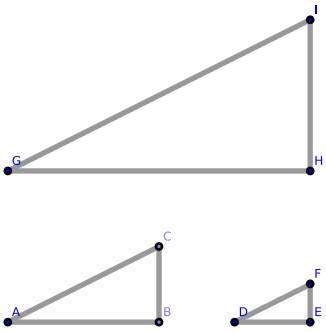


FIGURE 6.2.33. [6.21] Note that the polygons need not be triangles.

Since $ABC \sim GHI$, they are equiangular and have the sides about their equal angles proportional. Similarly, DEF and GHI are equiangular and have the sides about their equal angles proportional.

Hence ABC and DEF are equiangular and have the sides about their equal angles proportional; or, $ABC \sim DEF$. □

REMARK. Our proof did not use any properties of triangles that are absent in any arbitrary $n-$sided polygon.

COROLLARY. *6.21.1. Two similar polygons which are homothetic to a third are homothetic to one another.*

Exercises.

1. If three similar polygons are respectively homothetic, then their three centers of similitudes are collinear.

PROPOSITION 6.22. *PROPORTIONALITY OF FOUR SEGMENTS TO THE POLYGONS CONSTRUCTED UPON THEM.*

Four segments are proportional if and only if the rectilinear figures similar and similarly described upon them are also proportional.

PROOF. Construct similar polygons $\triangle ABK$ and $\triangle CDL$ constructed from $\overline{AB}$ and $\overline{CD}$ as well as $\square MEFI$ and $\square NGHJ$ constructed from $\overline{EF}$ and $\overline{GH}$.

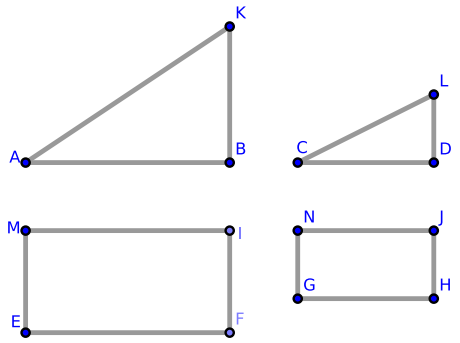


FIGURE 6.2.34. [6.22]

Suppose that $\overline{AB} : \overline{CD} = \overline{EF} : \overline{GH}$. We wish to show that

$$\triangle ABK : \triangle CDL = \square HEFI : \square NGHJ$$

Notice that:

$$\begin{array}{rcll} \overline{AB} : \overline{CD} & = & \overline{EF} : \overline{GH} & \text{(by hypothesis)} \\ \triangle ABK : \triangle CDL & = & (\overline{AB})^2 : (\overline{CD})^2 & \text{[6.20]} \\ \square HEFI : \square NGHJ & = & (\overline{EF})^2 : (\overline{GH})^2 & \text{[6.20]} \end{array}$$

Thus $(\overline{AB})^2 : (\overline{CD})^2 = (\overline{EF})^2 : (\overline{GH})^2$; it follows that $\triangle ABK : \triangle CDL = \square HEFI : \square NGHJ$. This proves our first claim.

Now suppose $\triangle ABK : \triangle CDL = \square HEFI : \square NGHJ$; by the above, we obtain $(\overline{AB})^2 : (\overline{CD})^2 = (\overline{EF})^2 : (\overline{GH})^2$. By [5.22, Cor. 1],

$$\overline{AB} : \overline{CD} = \overline{EF} : \overline{GH}$$

This proves our final claim. □

PROPOSITION 6.23. *EQUIANGULAR PARALLELOGRAMS.*

The areas of equiangular parallelograms have a ratio to each other equal to the ratio of the rectangles contained by their sides about a pair of equal angles.

PROOF. Construct equiangular parallelograms $\boxdot HABD$ and $\boxdot BGEC$ where $\angle ABD = \angle GBC$. We claim that

$$\boxdot HABD : \boxdot BGEC = \left(\overline{AB} \cdot \overline{BD}\right) : \left(\overline{BC} \cdot \overline{BG}\right)$$

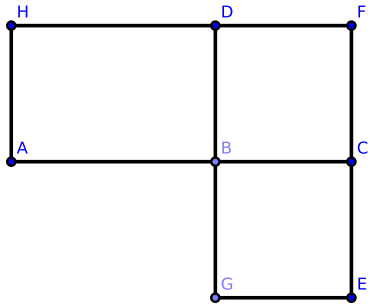


FIGURE 6.2.35. [6.23]

Let the sides $\overline{AB}$ and $\overline{BC}$ about the equal angles $\angle ABD$ and $\angle CBG$ be placed such that $\overline{AC} = \overline{AB} \oplus \overline{BC}$. As in [6.14], $\overline{GB} \oplus \overline{BD} = \overline{GD}$.

Complete the parallelogram $\boxdot DBCF$. By [6.1],

$$\begin{aligned} \boxdot HABD : \boxdot DBCF &= \overline{AB} : \overline{BC} \\ \boxdot DBCF : \boxdot BGEC &= \overline{BD} : \overline{BG} \end{aligned}$$

It follows that $\boxdot HABD : \boxdot BGEC = \left(\overline{AB} \cdot \overline{BD}\right) : \left(\overline{BC} \cdot \overline{BG}\right)$, which completes the proof. □

Exercises.

1. Triangles which have one angle of one equal or supplemental to one angle of the other are to one another in the ratio of the rectangles of the sides about those angles.

2. Two quadrilaterals whose diagonals intersect at equal angles are to one another in the ratio of the rectangles of the diagonals.

PROPOSITION 6.24. *SIMILAR PARALLELOGRAMS ABOUT THE DIAGONAL.*

In any parallelogram, the parallelograms about the diameter are similar both to the whole and to one another.

PROOF. Construct $\boxdot BADC$ such that $\overline{AC}$ is its diameter and where $\boxdot EAGF$ and $\boxdot HFKC$ are the parallelograms about $\overline{AC}$. We claim that $\boxdot BADC \sim \boxdot EAGF$, $\boxdot BADC \sim \boxdot HFKC$, and $\boxdot EAGF \sim \boxdot HFKC$.

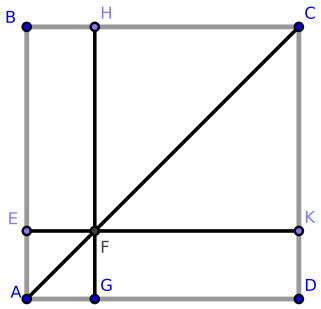


FIGURE 6.2.36. [6.24]

Consider $\triangle ABC$: since $\overline{EF} \parallel \overline{BC}$, by [6.2] $\overline{BE} : \overline{EA} = \overline{CF} : \overline{FA}$.

Consider $\triangle ACD$: since $\overline{FG} \parallel \overline{CD}$, also by [6.2] $\overline{CF} : \overline{FA} = \overline{DG} : \overline{GA}$.

By [5.18], it follows that $\overline{BE} : \overline{EA} = \overline{DG} : \overline{GA}$.

By [5.16], $\overline{BA} : \overline{EA} = \overline{DA} : \overline{GA}$, and so $\overline{BA} : \overline{DA} = \overline{EA} : \overline{GA}$. It follows that in $\boxdot BADC$ and $\boxdot EAGF$ the sides about the common angle $\angle BAD$ are proportional.

Consider $\triangle ADC$ and $\triangle AGF$: since $\overline{GF} \parallel \overline{DC}$, $\angle AFG = \angle ACD$; the triangles also share $\angle DAC = \angle GAF$ in common. Thus, $\triangle ADC$ and $\triangle AGF$ are equiangular by [1.29].

Similarly, $\triangle ACB$ and $\triangle AFE$ are equiangular; thus $\boxdot BADC$ and $\boxdot EAGF$ are equiangular.

This gives us:

$$\begin{aligned}
\overline{AD} : \overline{DC} &= \overline{AG} : \overline{GF} \\
\overline{DC} : \overline{CA} &= \overline{GF} : \overline{FA} \\
\overline{CA} : \overline{CB} &= \overline{FA} : \overline{FE} \\
\overline{CB} : \overline{BA} &= \overline{FE} : \overline{EA}
\end{aligned}$$

The second and third proportions give us: $\overline{DC} : \overline{CB} = \overline{GF} : \overline{FE}$.

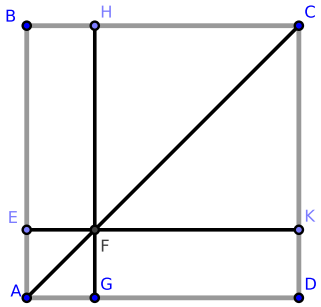


FIGURE 6.2.37. [6.24]

Therefore in $\boxdot BADC$ and $\boxdot EAGF$ the sides about the equal angles are proportional, and so $\boxdot BADC \sim \boxdot EAGF$.

Mutatis mutandis, $\boxdot BADC \sim \boxdot HFKC$. By transitivity, $\boxdot EAGF \sim \boxdot HFKC$.

This proves our claim. □

COROLLARY. *6.24.1. The parallelograms $\boxdot EAGF$, $\boxdot JFHC$, and $\boxdot BADC$ are, respectively, homothetic.*

PROPOSITION 6.25. *CONSTRUCTION OF A POLYGON EQUAL IN AREA TO A GIVEN FIGURE AND SIMILAR TO A SECOND GIVEN FIGURE.*

PROOF. We wish to construct a polygon equal in area to $ALMN$ and similar to polygon BCD.

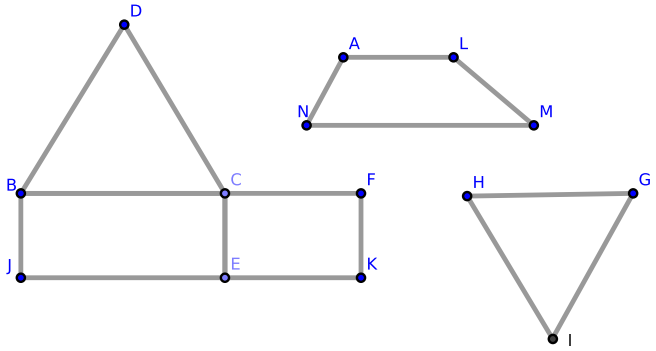


FIGURE 6.2.38. [6.25] Note that polygon BCD need not be a triangle.

Wlog, on side $\overline{BC}$ of the polygon BCD, construct the rectangle $\boxdot BJEC = \triangle BCD$ [1.44], and on $\overline{CE}$ construct the rectangle $\boxdot CEKF = ALMN$ [1.45].

Construct $\overline{GH}$ as the mean proportion between $\overline{BC}$ and $\overline{CF}$ [6.13]. On $\overline{GH}$, construct polygon $GHI \sim BCD$ [6.18] where $\overline{BC}$ and $\overline{GH}$ are corresponding sides. We claim that GHI is the required polygon.

We have $\overline{BC} : \overline{GH} = \overline{GH} : \overline{CF}$ by construction; by [Def 5.10], $\overline{BC} : \overline{CF}$ is in the duplicate ratio of $\overline{BC} : \overline{GH}$. Or,

$$\frac{BC}{CF} = \left(\frac{BC}{GH}\right)^2$$

Since $BCD \sim GHI$, by [6.20]

$$\frac{BCD}{GHI} = \left(\frac{BC}{GH}\right)^2$$

We also have that $\overline{BC} : \overline{CF} = \boxdot BJEC : \boxdot CEKF$, and so $\boxdot BJCE : \boxdot CEKF = BCD : GHI$.

But the rectangle $\boxdot BJEC$ is equal in area to the polygon BCD; therefore, $\boxdot CEKF = GHI$. Recall that by construction $\boxdot CEKF = ALMN$. It follows that $GHI = ALMN$ and is similar to BCD. This proves our claim. □

Alternatively:

PROOF. Construct squares $\boxdot EFJK$ and $\boxdot LMNO$ such that $\boxdot EFJK =$ polygon DCB and $\boxdot LMNO =$ polygon $APQS$ [2.14]. By [6.12], construct $\overline{GH}$, the fourth proportional to $\overline{EF}$, $\overline{LM}$, and $\overline{BC}$.

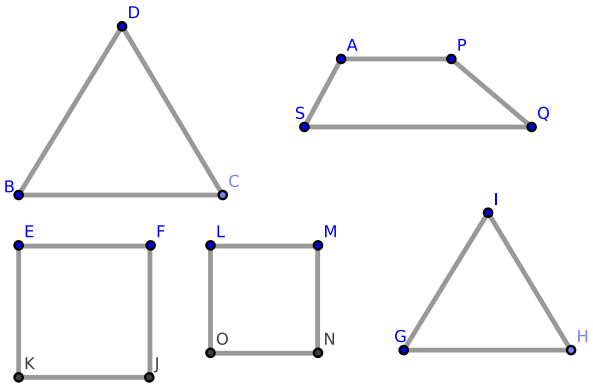


FIGURE 6.2.39. [6.25], alternate proof. Note that BCD need not be a triangle.

On $\overline{GH}$, construct the polygon GHI similar to the polygon BCD [6.18] such that $\overline{BC}$ and $\overline{GH}$ are corresponding sides. We claim that GHI is the required polygon.

Because $\overline{EF} : \overline{LM} = \overline{BC} : \overline{GH}$ by construction, we have that $EFJK : LMNO = BCD : GHI$ [6.22]. But $EFJK = BCD$ by construction; therefore, $LMNO = GHI$. But we also have that $LMNO = APQS$ by construction. Therefore $GHI = APQS$ and is similar to BCD. □

PROPOSITION 6.26. *PARALLELOGRAMS ON A COMMON ANGLE.*

If two similar and similarly situated parallelograms have a common angle, then they stand on the same diagonal.

PROOF. Construct $\boxdot AEFG$ and $\boxdot ABCD$ such that each are similar and similarly situated and have $\angle GAF$ as a common angle. We claim that they stand on the same diagonal, $\overline{AC}$.

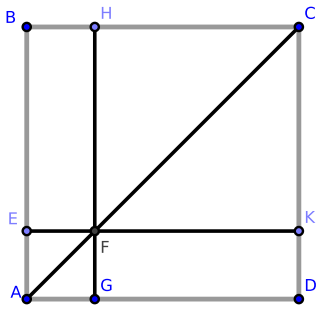


FIGURE 6.2.40. [6.26]

Construct the diagonals $\overline{AF}$ and $\overline{AC}$. Because $\boxdot AEFG \sim \boxdot ABCD$ by hypothesis, they can be divided into the same number of similar triangles by [6.20]. Hence, $\triangle GAF \sim \triangle CAD$, and it follows that $\angle GAF = \angle CAD$.

Hence, $\overline{AC}$ must pass through the point F, and so the parallelograms stand on the same diagonal. □

REMARK. [6.26] is the converse of [6.24] and may have been misplaced in an early edition of Euclid. The following would be a simpler statement of result: "If two homothetic parallelograms have a common angle, they are about the same diagonal."

PROPOSITION 6.27. *INSCRIBING A PARALLELOGRAM IN A TRIANGLE I.*

We wish to inscribe in a given triangle the parallelogram with maximum area having a common angle with a given triangle.

PROOF. Construct $\triangle ABC$ where the given angle is $\angle ABC$. Bisect $\overline{AC}$ at P. Through P, construct $\overleftrightarrow{PE} \parallel \overline{BC}$ and $\overleftrightarrow{PF} \parallel \overline{AB}$. We claim that $\boxdot EBFP$ is the required parallelogram.

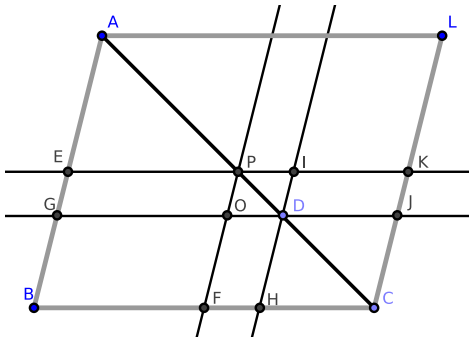


FIGURE 6.2.41. [6.27]

Construct $\overline{AL}$ such that $\overline{AL} = \overline{BC}$ and $\overline{AL} \parallel \overline{BC}$, and construct $\overline{CL}$ such that $\overline{CL} = \overline{BA}$ and $\overline{CL} \parallel \overline{BA}$.

Take any other point D on $\overline{AC}$, construct $\overleftrightarrow{DG} \parallel \overline{BC}$ which intersects $\overleftrightarrow{PF}$ at O, $\overline{AB}$ at G, and $\overline{CL}$ at J. Also construct $\overleftrightarrow{DH} \parallel \overline{AB}$ which intersects $\overleftrightarrow{EK}$ at I and $\overline{BC}$ at H.

Since $\overline{AC}$ is bisected at P, $\overline{EK}$ is also bisected in P. By [1.36], $\boxdot EGOP = \boxdot POJK$. Therefore, $\boxdot EGOP > \boxdot IDJK$; but $\boxdot IDJK = \boxdot OFHD$ [1.43], and so $\boxdot EGOP > \boxdot OFHD$.

Add $\boxdot GBFO$ to each, and we have that $\boxdot EBFP > \boxdot GBHD$. Since our choice of D was any point on $\overline{AC}$ other than P, and since $\boxdot EBFP$ contains $\angle ABC$, $\boxdot EBFP$ is the maximum parallelogram which can be inscribed in the triangle $\triangle ABC$ which contains $\angle ABC$. This proves our claim. □

COROLLARY. *6.27.1. The maximum parallelogram exceeds any other parallelogram about the same angle in the triangle by the area of the similar parallelogram whose diagonal is the line between the midpoint P of the opposite side and the point D, which is the corner of the other inscribed parallelogram.*

COROLLARY. *6.27.2. The parallelograms inscribed in a triangle and having one angle common with it are proportional to the rectangles contained by the segments of the sides of the triangle made by the opposite corners of the parallelograms.*

COROLLARY. *6.27.3. The parallelogram in [6.27] has proportions* $\overline{AC} : \overline{GH} = (\overline{AC})^2 : \big(\overline{AD} \cdot \overline{DC}\big)$.

Exercises.

1. Prove [Cor. 6.27.1].
2. Prove [Cor. 6.27.2].
3. Prove [Cor. 6.27.3].

PROPOSITION 6.28. *INSCRIBING A PARALLELOGRAM IN A TRIANGLE II.*

We wish to inscribe in a given triangle a parallelogram equal in area to a given polygon (whose area is not greater than the maximum inscribed parallelogram within the triangle) and which has an angle common with the triangle.

PROOF. Construct $\triangle ABC$ and polygon XYZ. In $\triangle ABC$, construct its maximum parallelogram $\boxdot EPFB$ as in [6.27]. We wish to inscribe a parallelogram in $\triangle ABC$ equal in area to XYZ (where $XYZ \leq \boxdot EBFP$) and which contains $\angle ABC$.

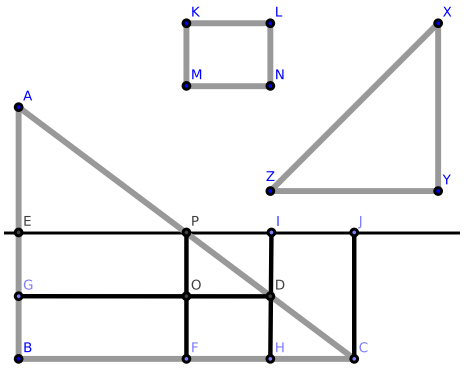


FIGURE 6.2.42. [6.28] Note that XYZ need not be a triangle.

Construct $\boxdot EPFB$ in $\triangle ABC$ as in [6.27] where $\boxdot EPFB$ is the maximum parallelogram that can be inscribed in $\triangle ABC$. If $XYZ = \boxdot EPFB$, the proof is complete.

Otherwise, extend $\overline{EP}$ to $\overrightarrow{EP}$, and construct $\overline{JC} \parallel \overline{PF}$ such that $\overline{JC} = \overline{PF}$. Also construct $\boxdot KLMN$ such that:

(1) $\boxdot KLMN$ equals the difference of the areas between the polygons $\boxdot PJCF$ and XYZ [6.25], and

(2) $\boxdot KLMN \sim \boxdot PJCF$ where $\overline{PJ}$ and $\overline{KL}$ correspond.

Between P and J, construct $\overline{PI} = \overline{KL}$. Construct $\overline{IH} \parallel \overline{AB}$ such that $\overline{IH} = \overline{PF}$, intersecting $\overline{AC}$ at D. Also construct $\overline{DG} \parallel \overline{BC}$ where G is a point on $\overline{AB}$. We claim that $\boxdot GDHB$ is the required parallelogram.

Since $\boxdot PJCF$ and $\boxdot PIDO$ stand on the same diagonal, $\boxdot PJCF \sim \boxdot PIDO$ by [6.24]; however, $\boxdot PJCF \sim \boxdot KLMN$ by construction, and by [6.21] $\boxdot PIDO \sim \boxdot KLMN$.

Since $\overline{PI} = \overline{KL}$, and since they are the corresponding sides of the parallelograms, by [6.20] $\boxdot PIDO = \boxdot KLMN$.

By [6.27, Cor. 1], $\boxdot PIDO$ is the difference between $\boxdot GDHB$ and $\boxdot EPFB$, and $\boxdot KLMN$ is the difference between $\boxdot PJCF$ and XYZ by construction. Therefore, the difference between $\boxdot PJCF$ and XYZ equals the difference between $\boxdot GDHB$ and $\boxdot EPFB$.

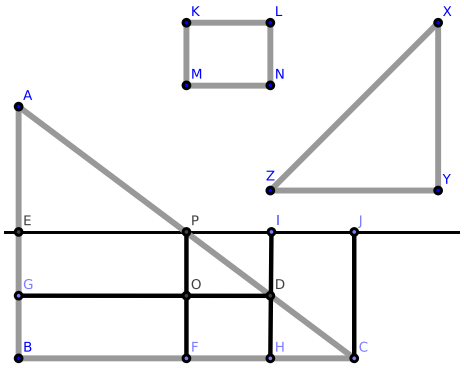


FIGURE 6.2.43. [6.28] Note that XYZ need not be a triangle.

However, $\boxdot EBFP = \boxdot PJCF$; therefore, $\boxdot GBHD = XYZ$. This proves our claim. □

PROPOSITION 6.29. *ESCRIBING A PARALLELOGRAM TO A TRIANGLE.*

We begin with three objects: a polygon, a segment, and a parallelogram. We wish to construct a new parallelogram constructed on the given segment which is equal in area to the given polygon and and then exceed that area by a parallelogram similar to the given parallelogram.

PROOF. Let $CUVWX$ be the given polygon, $\overline{AB}$ be the given segment, and $\boxdot DRST$ the parallelogram to which the excess area is required to be similar.

We wish to construct a parallelogram equal in area to $CUVWX$ on the segment $\overline{AB}$ and then exceeding that area by the area of a parallelogram similar to $\boxdot DRST$. Specifically, we claim that $\boxdot APOI = CUVWX$.

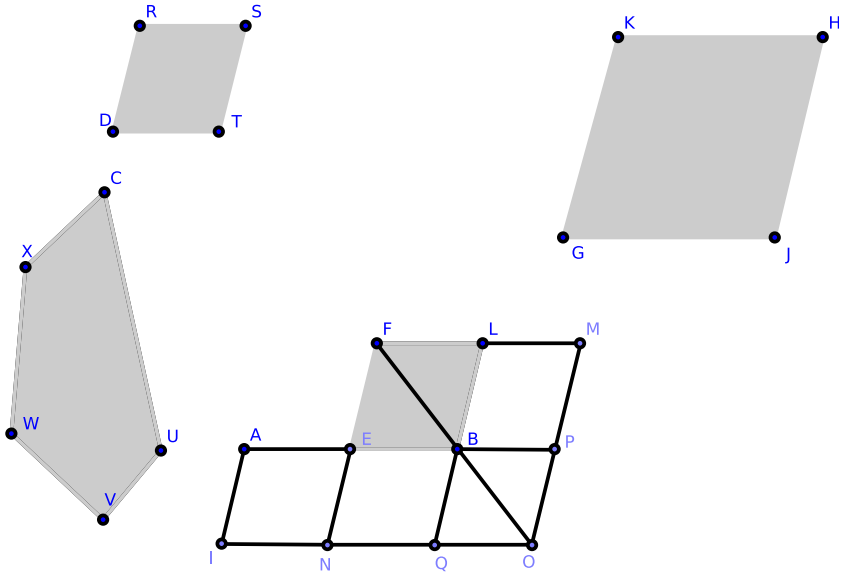


FIGURE 6.2.44. [6.29]

Bisect $\overline{AB}$ at E. Construct $\boxdot FEBL$ on $\overline{EB}$ similar and similarly situated to $\boxdot DRST$, and by [6.25] construct $\boxdot GKHJ = \boxdot FEBL + CUVWX$ which is similar and similarly situated to $\boxdot DRST$. Let $\overline{KH}$ correspond to $\overline{FL}$ and $\overline{KG}$ correspond to $\overline{FE}$.

Since $\boxdot GKHJ > \boxdot FEBL$, it follows that $\overline{KH} > \overline{FL}$ and $\overline{KG} > \overline{FE}$.

Extend $\overline{FL}$ to $\overline{FM}$ where $\overline{FM} = \overline{KH}$, and extend $\overline{FE}$ to $\overline{FN}$ where $\overline{FN} = \overline{KG}$.

Construct $\boxdot FMON$; then $\boxdot FMON = \boxdot GKHJ$ and $\boxdot FMON \sim \boxdot GKHJ$ by [6.26].

But $\boxdot GKHJ \sim \boxdot FEBL$, and so $\boxdot FMON \sim \boxdot FEBL$ [6.21]; it follows that $\boxdot FMON$ and $\boxdot FEBL$ stand on the same diameter.

Construct diameter $\overline{FO}$ of $\boxdot FMON$. Since $\boxdot GKHJ = \boxdot FEBL + CUVWX$ and $\boxdot GKHJ = \boxdot FMON$, it follows that $\boxdot FMON = \boxdot FEBL + CUVWX$.

Subtract the area of $\boxdot FEBL$ from each side of the equation: hence the gnomon $EBLMON = CUVWX$.

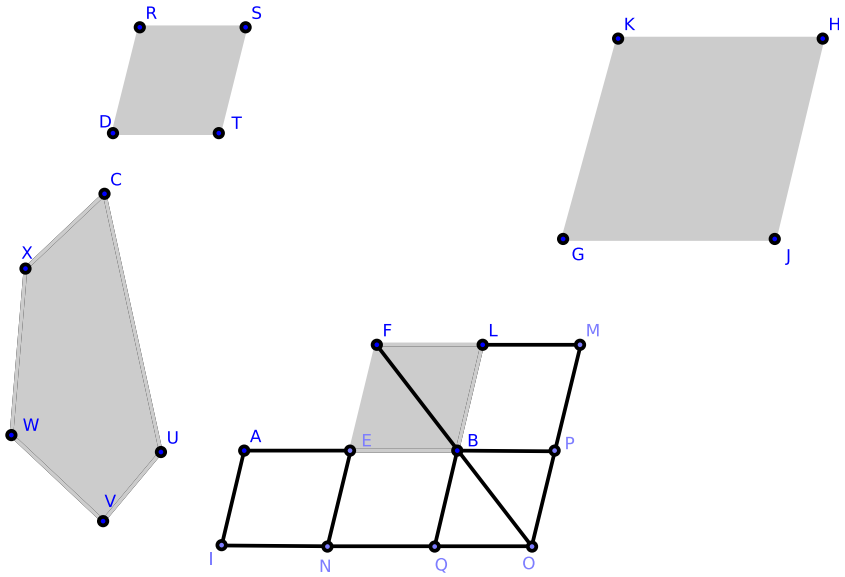


FIGURE 6.2.45. [6.29]

Construct $\overline{OI}$ such that $\overline{OI} \parallel \overline{FM}$ and $\overline{OI} = \overline{FM} + \overline{AE}$. Also construct $\overline{BQ} \parallel \overline{EN}$ where $\overline{BQ} = \overline{EN}$. Finally, construct $\overline{BP}$ such that $\overline{BP} \parallel \overline{LM}$ and $\overline{BP} = \overline{LM}$.

Since $\overline{AE} = \overline{EB}$, it follows that $\boxdot AENI = \boxdot EBQN = \boxdot LMBP$; and

$$\begin{aligned} \boxdot AENI + \boxdot EPON &= \boxdot LMBP + \boxdot EPON \\ \boxdot APOI &= EBLMON \end{aligned}$$

By the above, $EBLMON = CUVWX$, and so $\boxdot APOI = CUVWX$. This proves our claim. □

PROPOSITION 6.30. *EXTREME AND MEAN RATIO OF A SEGMENT.*

We wish to divide a given segment into its "extreme and mean ratio."

PROOF. We wish to divide $\overline{AB}$ into its "extreme and mean ratio."

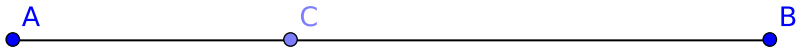


FIGURE 6.2.46. [6.30]

Divide $\overline{AB}$ at C so that the rectangle $\overline{AB} \cdot \overline{BC} = (\overline{AC})^2$ [2.11]. We claim that C is the required point.

Because the rectangle $\overline{AB} \cdot \overline{BC} = (\overline{AC})^2$, we have that $\overline{AB} : \overline{AC} = \overline{AC} : \overline{BC}$ [6.17]. By [Def. 6.2], $\overline{AB}$ is cut in extreme and mean ratio at C. □

Exercises.

1. If the three sides of a right triangle are in continued proportion, prove that the hypotenuse is divided in extreme and mean ratio by the perpendicular from the right angle on the hypotenuse.

2. In the same case as #1, the greater segment of the hypotenuse is equal to the least side of the triangle.

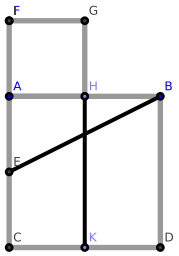


FIGURE 6.2.47. [6.30], #3

3. Prove that the square on the diameter of the circle constructed about the triangle formed by the points F, H, D is equal to six times the square on the segment $\overline{FD}$.

PROPOSITION 6.31. *AREA OF SQUARES ON A RIGHT TRIANGLE.*

If any similar quadrilateral is similarly constructed on the three sides of a right triangle, the quadrilateral on the hypotenuse is equal in area to the sum of the areas of the quadrilaterals constructed on the two other sides.

PROOF. Denote the sides by a, b, and c and the areas of the polygons by α, β, and γ. We claim that $\alpha + \beta = \gamma$.

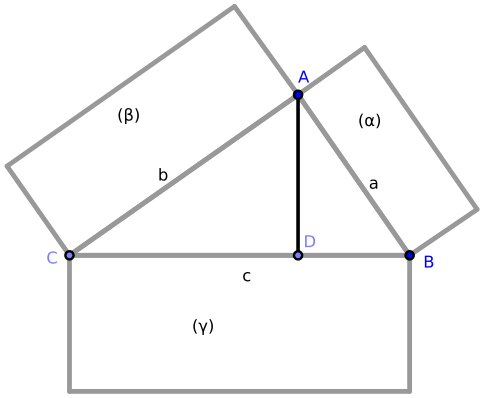


FIGURE 6.2.48. [6.31]

Because the polygons are similar, by [6.20] we have that $\alpha : \gamma = a^2 : c^2$.

Therefore,

$$\frac{\alpha}{\gamma} = \frac{a^2}{c^2}$$

Similarly,

$$\frac{\beta}{\gamma} = \frac{b^2}{c^2}$$

It follows that

$$\frac{\alpha + \beta}{\gamma} = \frac{a^2 + b^2}{c^2}$$

But $a^2 + b^2 = c^2$ by [1.47]. Therefore, $\alpha + \beta = \gamma$; this proves our claim. □

Exercise.

1. If semicircles are constructed on supplemental chords of a semicircle, the sum of the areas of the two crescents thus formed is equal to the area of the triangle whose sides are the supplemental chords and the diameter.

PROPOSITION 6.32. *FORMATION OF TRIANGLES.*

If two triangles having two sides proportional to two sides are placed together at one angle so that their corresponding sides are also parallel, then the remaining sides of the triangles are in a straight line.

PROOF. Construct $\triangle ABC$ and $\triangle DCE$ such that $\overline{AB} : \overline{AC} = \overline{DC} : \overline{DE}$, $\overline{AB} \parallel \overline{DC}$, and $\overline{AC} \parallel \overline{DE}$. We claim that $\overline{BC} \oplus \overline{CE} = \overline{BE}$.

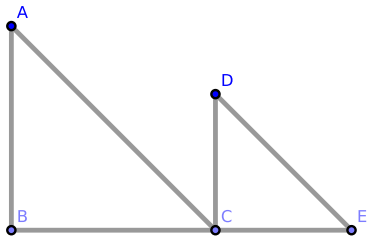


FIGURE 6.2.49. [6.32]

Since $\overline{AB} \parallel \overline{DC}$ and each intersects $\overline{AC}$, we have that $\angle BAC = \angle ACD$. Similarly, $\angle CDE = \angle ACD$; hence, $\angle BAC = \angle CDE$.

Consider $\triangle ABC$ and $\triangle DCE$: $\angle BAC = \angle CDE$ and $\overline{AB} : \overline{AC} = \overline{DC} : \overline{DE}$. By [6.6], $\triangle ABC$ and $\triangle DCE$ are equiangular. So, $\angle ABC = \angle DCE$.

Furthermore,

$$\begin{aligned} \angle ACE &= \angle ACD + \angle DCE \\ &= \angle BAC + \angle ABC \\ &\Rightarrow \\ \angle ACE + \angle ACB &= \angle BAC + \angle ABC + \angle ACB \end{aligned}$$

Since $\angle BAC + \angle ABC + \angle ACB$ equals two right angles by [1.32], so does $\angle ACE + \angle ACB$. Thus, $\overline{BE} = \overline{BC} \oplus \overline{CE}$, which proves our claim. □

PROPOSITION 6.33. *RATIOS OF EQUAL TRIANGLES.*

In equal circles, angles at the centers or at the circumferences have the same ratio to one another as the arcs on which they stand. This also holds true for sectors.

PROOF. Construct equal circles $\bigcirc O$ and $\bigcirc P$. We claim that the angles at the centers ($\angle BOC$, $\angle EPF$) or at the circumferences ($\angle BAC$, $\angle EDF$) have the same ratio to one another as the arcs (BC, EF) on which they stand. This also holds true for sectors (BOC, EPF).

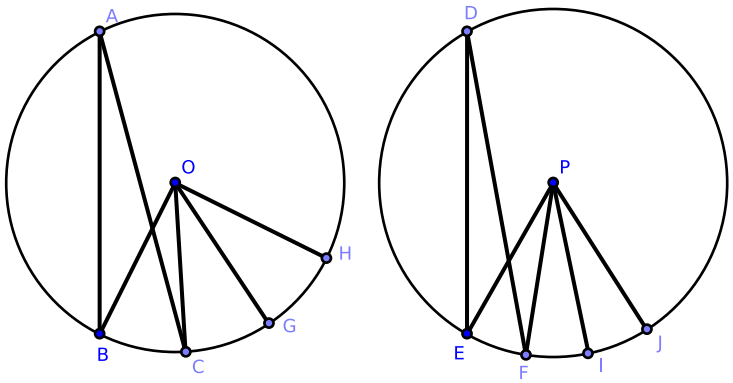


FIGURE 6.2.50. [6.33]

We prove each claim separately.

Claim 1: $BC : EF = \angle BAC : \angle EDF$ where BC and EF are arcs.

Construct any number of arcs in the first circle equal in length to BC; in this proof, we construct CG and GH. In $\bigcirc P$, Construct any finite number of arcs equal to EF; we construct FI and IJ. Also construct $\overline{OG}$, $\overline{OH}$, $\overline{IP}$, and $\overline{JP}$.

Because $BC = CG = GH$, by [3.27] $\angle BOC = \angle COG = \angle GOH$. Therefore, there exists some positive integer n such that $BH = n \cdot BC$ and $\angle BOH = n \cdot \angle BOC$.

Similarly, $EJ = n \cdot EF$ and $\angle EPJ = n \cdot \angle EPF$.

And since the circles are equal, it is evident that $\angle BOH$ is either greater than, equal to, or less than $\angle EPJ$ according to whether the arc BH is greater than, equal to, or less than the arc EJ.

Assume $EF < BC$; then $BC = k \cdot EF$ where k is a positive real number. It follows that $\angle BOC = k \cdot \angle EPF$, and so $BC : EF = \angle BOC : \angle EPF$.

Since by [3.20] $\angle BAC = \frac{1}{2} \cdot \angle BOC$ and $\angle EDF = \frac{1}{2} \cdot \angle EPF$,

$$\angle BOC : \angle EPF \quad = \quad \angle BAC : \angle EDF \quad \text{[5.15]}$$

from which it follows that $BC : EF = \angle BAC : \angle EDF$ [5.11]. This proves our claim.

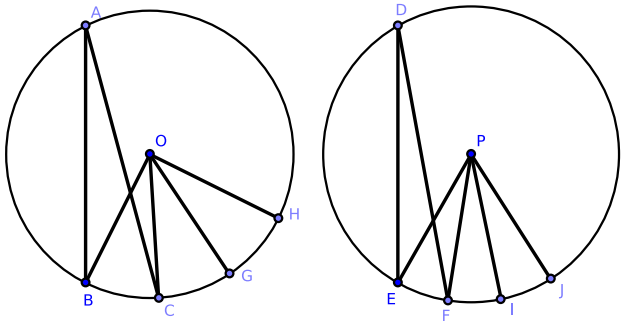


FIGURE 6.2.51. [6.33]

Claim 2: sector BOC : sector $EPF = BC : EF$.

Since the circles are equal, and since the arc BC is equal in length to CG, $\angle BOC = \angle COG$. Hence the sectors BOC, COG are congruent (see Observation, [3.29]); therefore, they are equal in area.

Similarly, the sectors COG, GOH are equal in area. Hence there are as many equal sectors as there are equal arcs; therefore, the arc BH and the sector BOH are equimultiples of the arc BC and the sector BOC.

Similarly, it may be shown that the arc EJ and the sector EPJ are equimultiples of the arc EF and the sector EPF. And it is evident by superposition that if the arc BH is greater than, equal to, or less than the arc EJ, the sector BOH is greater than, equal to, or less than the sector EPJ. Hence by [Def. 5.5], arc BC : arc EF = sector BOC : sector EPF.

This completes our claim. □

Exam questions for chapter 6.

1. What is the subject-matter of chapter 6? (Ans. Application of the theory of proportion.)

2. What are similar polygons?

3. What do similar polygons agree in?

4. How many conditions are necessary to define similar triangles?

5. How many conditions are necessary to define similar polygons of more than three sides?

6. When is a polygon said to be given in species?

7. When in magnitude?

8. When in position?

9. What is a mean proportional between two lines?

10. Define two mean proportionals.

11. What is the altitude of a polygon?

12. If two triangles have equal altitudes, how do their areas vary?

13. How do these areas vary if they have equal bases but unequal altitudes?

14. If both bases and altitudes differ, how do the areas vary?

15. When are two lines divided proportionally?

16. If in two segments which are divided proportionally a pair of corresponding points coincide with their point of intersection, what property holds for the lines joining the other pairs of corresponding points?

17. Define "reciprocal proportion".

18. If two triangles have equal areas, prove that their perpendiculars are reciprocally proportional to the bases.

19. What is meant by inversely similar polygons?

20. If two polygons are inversely similar, how can they be changed into polygons which are directly similar?

21. Give an example of two triangles inversely similar. (Ans. If two lines passing through any point O outside a circle intersect it in pairs of points A, A'; B, B', respectively, the triangles $\triangle OAB$, $\triangle OA'B'$ are inversely similar.)

22. What point is it round which a polygon can be turned so as to bring its sides into positions of parallelism with the sides of a similar polygon? (Ans. The center of similitude of the two polygons.)

23. How many polygons similar to a given polygon of sides can be constructed on a given line?

24. How many centers of similitude can two regular polygons of n sides each have? (Ans. n centers, which lie on a circle.)

25. What are homothetic polygons?

26. How do the areas of similar polygons vary?

27. What proposition is [6.19] a special case of?

28. Define Philo's line.

29. How many centers of similitude do two circles have?

Exercises for chapter 6.

1. If in a fixed triangle, we construct a variable side parallel to the base, the locus of the points of intersection of the diagonals of the trapezoid that is cut off from the triangle is the median that bisects the base.

2. Find the locus of the point which divides in a given ratio the several lines constructed from a given point to the circumference of a given circle.

3. Two segments $\overline{AB}$, $\overline{XY}$ are given in position where $\overline{AB}$ is divided at C in the ratio $m : n$ and parallels $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ are constructed in any direction meeting $\overline{XY}$ in the points A', B', C'. Prove that

$$(m+n)\cdot\overline{CC'} = n\cdot\overline{AA'} + m\cdot\overline{BB'}$$

4. Three concurrent lines from the vertices of a triangle $\triangle ABC$ meet the opposite sides in A', B', C'. Prove $\overline{AB'}\cdot\overline{BC'}\cdot\overline{CA'} = \overline{A'B}\cdot\overline{B'C}\cdot\overline{C'A}$.

5. If a transversal meets the sides of a triangle $\triangle ABC$ at the points A', B', C', prove $\overline{AB'}\cdot\overline{BC'}\cdot\overline{CA'} = -\overline{A'B}\cdot\overline{B'C}\cdot\overline{C'A}$.

6. If on a variable segment $\overline{AC}$ which is constructed from a fixed point A to any point B on the circumference of a given circle, a point C is taken such that the rectangle $\overline{AB}\cdot\overline{AC}$ is constant, prove that the locus of C is a circle.

7. If D is the midpoint of the base BC of a triangle $\triangle ABC$, E the foot of the perpendicular, L is the point where the bisector of the angle at A meets BC, and H the point of intersection of the inscribed circle with BC, prove that $\overline{DE}\cdot\overline{HL} = \overline{HE}\cdot\overline{HD}$.

8. As in #7, if K is the point of intersection with $\overline{BC}$ of the escribed circle, which touches the other extended sides, prove that $\overline{LH}\cdot\overline{BK} = \overline{BD}\cdot\overline{LE}$.

9. If R, r, r', r'', r''' are the radii of the circumscribed, the inscribed, and the escribed circles of a plane triangle, d, d', d'', d''' the distances of the center of the circumscribed circle from the centers of the others, then $R^2 = d^2 + 2Rr = d'^2 - 2Rr'$, etc.

10. As in #9, prove that $12R^2 = d^2 + d'^2 + d''^2 + d'''^2$.

11. If p', p'', p''' denote the altitudes of a triangle, then:

$$\begin{array}{lrcll} (1) & \frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''} & = & \frac{1}{r} & \\ (2) & \frac{1}{p''} + \frac{1}{p'''} - \frac{1}{p} & = & \frac{1}{r'} & \text{(etc.)} \\ (3) & \frac{2}{p} & = & \frac{1}{r} - \frac{1}{r'} & \text{(etc.)} \\ (4) & \frac{2}{p'} & = & \frac{1}{r''} + \frac{1}{r'''} & \text{(etc.)} \end{array}$$

12. In a given triangle, inscribe another of given form which has one of its angles at a given point in one of the sides of the original triangle.

13. If a triangle of given form moves so that its three sides pass through three fixed points, the locus of any point in its plane is a circle.

14. Suppose that the angle at point A and the area of a triangle $\triangle ABC$ are given in magnitude. If the point A is fixed in position and the point B move along a fixed line or circle, then the locus of the point C is a circle.

15. One of the vertices of a triangle of given form remains fixed, and the locus of another is a segment or circle. Find the locus of the third.

16. Find the area of a triangle:

(a) in terms of its medians;

(b) in terms of its perpendiculars.

17. If two circles touch externally, their common tangent is a mean proportional between their diameters.

18. If there are three given parallel lines and two fixed points A, B, and if the lines connecting A and B to any variable point in one of the parallels intersects the other parallels at the points C and D, E and F, respectively, prove that $\overline{CF}$ and $\overline{DE}$ each pass through a fixed point.

19. If a system of circles pass through two fixed points, any two secants passing through one of the points are cut proportionally by the circles.

20. Find a point O in the plane of a triangle $\triangle ABC$ such that the diameters of the three circles about the triangles $\triangle OAB$, $\triangle OBC$, $\triangle OCA$ may be in the ratios of three given segments.

21. Suppose that $ABCD$ is a cyclic quadrilateral, and the segments $\overline{AB}$, $\overline{AD}$, and the point C are given in position. Find the locus of the point which divides $\overline{BD}$ in a given ratio.

22. If $\overline{CA}$, $\overline{CB}$ are two tangents to a circle and $\overline{BE} \perp \overline{AD}$ (where $\overline{AD}$ is the the diameter through A), then prove that $\overline{CD}$ bisects $\overline{BE}$.

23. If three segments from the vertices of a triangle $\triangle ABC$ to any interior point O meet the opposite sides in the points A', B', C', prove that

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1$$

24. If three concurrent segments OA, OB, OC are cut by two transversals in the two systems of points A, B, C; A', B', C', respectively, then prove that

$$\frac{AB}{A'B'} \cdot \frac{OC}{OC'} = \frac{BC}{B'C'} \cdot \frac{OA}{OA'} = \frac{CA}{C'A'} \cdot \frac{OB}{OB'}$$

25. The line joining the midpoints of the diagonals of a quadrilateral circumscribed to a circle:

(a) divides each pair of opposite sides into inversely proportional segments;

(b) is divided by each pair of opposite segments into segments which when measured from the center are proportional to the sides;

(c) is divided by both pairs of opposite sides into segments which when measured from either diagonal have the same ratio to each other.

26. If $\overline{CD}$, $\overline{CD'}$ are the internal and external bisectors of the angle at C of the triangle $\triangle ACB$, the three rectangles $\overline{AD} \cdot \overline{DB}$, $\overline{AC} \cdot \overline{CB}$, $\overline{AD} \cdot \overline{BD}$ are proportional to the squares of $\overline{AD}$, $\overline{AC}$, $\overline{AD}$ and are:

(a) in arithmetical progression, if the difference of the base angles is equal to a right angle;

(b) in geometrical progression if one base angle is right;

(c) in harmonic progression if the sum of the base angles is equal to a right angle.

27. If a variable circle touches two fixed circles, the chord of contact passes through a fixed point on the line connecting the centers of the fixed circles.

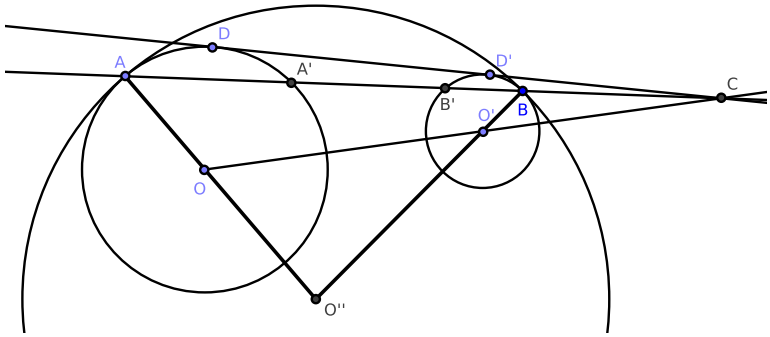


FIGURE 6.2.52. Ch. 6, #27

Let O,O' be the centers of the two fixed circles where O is the center of the variable circle. Let A, B the points of contact, and let $\overline{AB}$ and $\overline{OO'}$ meet at C, and cut the fixed circles again in the points A', B' respectively.

Construct $\overline{A'O}$, $\overline{AO}$, $\overline{BO'}$. Then $\overline{AO}$, $\overline{BO'}$ meet at O'' [3.11]. Now because $\triangle OAA'$, $\triangle O''AB$ are isosceles, the angles $\angle O''BA = \angle O''AB = \angle OA'A$.

Hence $\overline{OA'} \parallel \overline{O'B}$; therefore $\overline{OC} : \overline{O'C} = \overline{OA'} : \overline{O'B}$ is in a given ratio. Hence, C is a given point.

28. If $\overline{DD'}$ is the common tangent to the two circles, then $(\overline{DD'})^2 = \overline{AB'} \cdot \overline{A'B}$.

29. If R denotes the radius of O'' and ρ, ρ' the radii of O, O', then $(\overline{DD'})^2 : (\overline{AB})^2 = (R \pm \rho)(R \pm \rho') : R^2$ where the choice of sign depends on the nature of the contacts. (This result follows from #28.)

30. If four circles are tangential to a fifth, and if we denote by $\overline{12}$ the common tangent to the first and second, etc., then $\overline{12}.\overline{34} + \overline{23}.\overline{14} = \overline{13}.\overline{24}$.

31. The inscribed and escribed circles of any triangle are all touched by its nine-points circle.

32. The four triangles which are determined by four points, taken three by three, are such that their nine-points circles have one common point.

33. If a, b, c, d denote the four sides, and D, D' the diagonals of a quadrilateral, prove that the sides of the triangle, formed by joining the feet of the perpendiculars from any of its vertices on the sides of the triangle formed by the three remaining points, are proportional to the three rectangles ac, bd, DD'.

34. Prove the converse of Ptolemy's theorem (see [6.17], #10).

35. Construct a circle which:

(a) passes through a given point, and touch two given circles;

(b) touches three given circles.

36. If a variable circle touches two fixed circles, the tangent to it from their center of similitude through which the chord of contact passes is of constant length. (See #27 above.)

37. If segments $\overline{AD}$, $\overline{BD'}$ are extended, they meet at a point on the circumference of O'', and the line $O''P$ is perpendicular to DD'. (See #27 above.)

38. If A, B are two fixed points on two lines given in position, and A', B' are two variable points such that the ratio $\overline{AA'} : \overline{BB'}$ is constant, the locus of the point dividing $\overline{A'B'}$ in a given ratio is a segment.

39. If a segment $\overline{EF}$ divides proportionally two opposite sides of a quadrilateral, and a segment $\overline{GH}$ the other sides, each of these is divided by the other in the same ratio as the sides which determine them.

40. In a given circle, inscribe a triangle such that the triangle whose vertices are the feet of the perpendiculars from the endpoints of the base on the bisector of the vertical angle and the foot of the perpendicular from the vertical angle on the base may be a maximum.

41. In a circle, the point of intersection of the diagonals of any inscribed quadrilateral coincides with the point of intersection of the diagonals of the circumscribed quadrilateral whose sides touch the circle at the vertices of the inscribed quadrilateral.

42. Through two given points describe a circle whose common chord with another given circle may be parallel to a given line, or pass through a given point.

43. Being given the center of a circle, describe it so as to cut the legs of a given angle along a chord parallel to a given line.

44. If concurrent lines constructed from the angles of a polygon of an odd number of sides divide the opposite sides each into two segments, the product of one set of alternate segments is equal in area to the product of the other set.

45. If a triangle is constructed about a circle, the lines from the points of contact of its sides with the circle to the opposite vertices are concurrent.

46. If a triangle is inscribed in a circle, the tangents to the circle at its three vertices meet the three opposite sides at three collinear points.

47. The external bisectors of the angles of a triangle meet the opposite sides in three collinear points.

48. Construct a circle touching a given line at a given point and cutting a given circle at a given angle.

Definition: 2. The center of mean position of any number of points A, B, C, D, etc., is a point which may be found as follows: bisect the segment joining any two points A, B at G. Join G to a third point C; divide $\overline{GC}$ at H so that $\overline{GH} = \frac{1}{3} \cdot \overline{GC}$. Join H to a fourth point D and divide $\overline{HD}$ at K, so that $\overline{HK} = \frac{1}{4} \cdot \overline{HD}$, and so on. The last point found will be the center of mean position of the given points.

49. The center of mean position of the vertices of a regular polygon is the center of figure of the polygon.

50. The sum of the perpendiculars let fall from any system of points A, B, C, D, etc., whose number is n on any line L, is equal to n times the perpendicular from the center of mean position on L.

51. The sum of the squares of segments constructed from any system of points A, B, C, D, etc., to any point P exceeds the sum of the squares of segments from the same points to their center of mean position, O, by $n \cdot (\overline{OP})^2$.

52. If a point is taken within a triangle so as to be the center of mean position of the feet of the perpendiculars constructed from it to the sides of the triangle, the sum of the squares of the perpendiculars is a minimum.

53. Construct a quadrilateral being given two opposite angles, the diagonals, and the angle between the diagonals.

54. A circle rolls inside another of double its diameter; find the locus of a fixed point in its circumference.

55. Two points, C, D in the circumference of a given circle are on the same side of a given diameter. Find a point P in the circumference at the other side of the given diameter, $\overline{AB}$, such that $\overline{PC}$, $\overline{PD}$ may cut $\overline{AB}$ at equal distances from the center.

56. If the sides of any polygon be cut by a transversal, the product of one set of alternate segments is equal to the product of the remaining set.

57. A transversal being constructed cutting the sides of a triangle, the lines from the angles of the triangle to the midpoints of the segments of the transversal intercepted by those angles meet the opposite sides in collinear points.

58. If segments are constructed from any point P to the angles of a triangle, the perpendiculars at P to these segments meet the opposite sides of the triangle at three collinear points.

59. Divide a given semicircle into two parts by a perpendicular to the diameter so that the radii of the circles inscribed ob them may have a given ratio.

60. From a point within a triangle, suppose that perpendiculars fall on the sides; find the locus of the point when the sum of the squares of the lines joining the feet of the perpendiculars is given.

61. If a circle makes given intercepts on two fixed lines, the rectangle contained by the perpendiculars from its center on the bisectors of the angle formed by the lines is given.

62. If the base and the difference of the base angles of a triangle are given, the rectangle contained by the perpendiculars from the vertex on two lines through the midpoint of the base, parallel to the internal and external bisectors of the vertical angle, is constant.

63. The rectangle contained by the perpendiculars from the endpoints of the base of a triangle on the internal bisector of the vertical angle is equal to the rectangle contained by the external bisector and the perpendicular from the middle of the base on the internal bisector.

64. State and prove the corresponding theorem for perpendiculars on the external bisector.

65. Suppose that R, R' denote the radii of the circles inscribed in the triangles into which a right triangle is divided by the perpendicular from the right angle on the hypotenuse. If c is the hypotenuse and s is the semi-perimeter, $R^2 + R'^2 = (s - c)^2$.

66. If A, B, C, D are four collinear points, find a point O in the same line with them such that $\overline{OA} \cdot \overline{OD} = \overline{OB} \cdot \overline{OC}$.

67. Suppose the four sides of a cyclic quadrilateral are given; construct it.

68. Being given two circles, find the locus of a point such that tangents from it to the circles may have a given ratio.

69. If four points A, B, C, D are collinear, find the locus of the point P at which $\overline{AB}$ and $\overline{CD}$ stand opposite equal angles.

70. If a circle touches internally two sides of a triangle, $\overline{CA}$, $\overline{CB}$, and its circumscribed circle, the distance from C to the point of intersection on either side is a fourth proportional to the semi-perimeter, $\overline{CA}$ and $\overline{CB}$.

71. State and prove the corresponding theorem for a circle touching the circumscribed circle externally and two extended sides.

72. Pascal's Theorem: if the opposite sides of an irregular hexagon $ABCDEF$ inscribed in a circle are extended until they meet, the three points of intersection G, H, I are collinear. See Fig. 6.2.51.

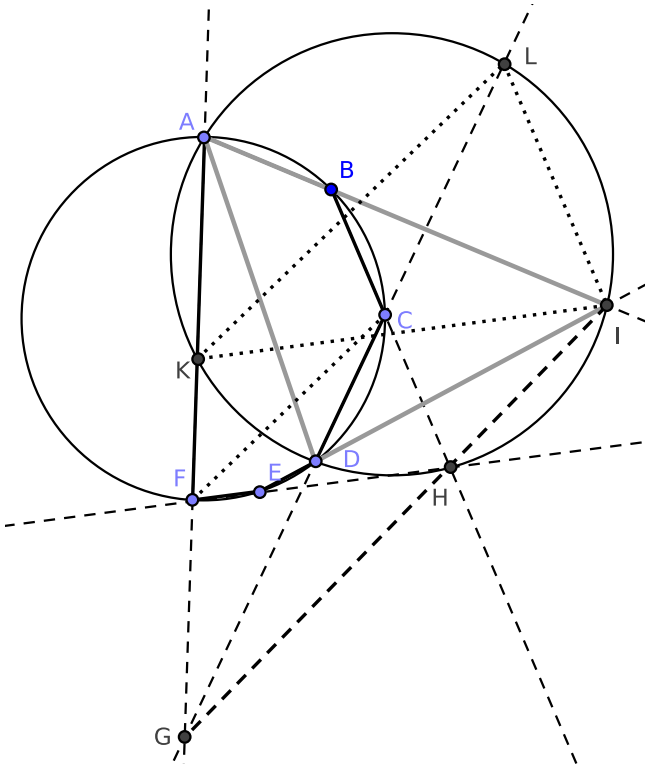


FIGURE 6.2.53. Ch. 6, #72. Pascal's Theorem

Construct $\overline{AD}$. Construct a circle about the triangle $\triangle ADI$, cutting the extended segments $\overline{AF}$, $\overline{CD}$, if necessary, at K and L. Construct $\overline{IK}$, $\overline{KL}$, $\overline{LI}$. By [3.21], we have that $\angle KLG = \angle FCG = \angle GAD$. Therefore $KL \parallel CF$.

Similarly, $\overline{LI} \parallel \overline{CH}$ and $\overline{KI} \parallel \overline{FH}$; hence the triangles $\triangle KLI$, $\triangle FCH$ are homothetic, and so the lines joining corresponding vertices are concurrent. Therefore, the points I, H, G are collinear.

73. If two sides of a triangle circumscribed to a given circle are given in position with the third side variable, the circle constructed about the triangle touches a fixed circle.

74. If two sides of a triangle are given in position, and if the area is given in magnitude, two points can be found at each of which the base stands opposite a constant angle.

75. If a, b, c, d denote the sides of a cyclic quadrilateral and s its semi-perimeter, prove that its area $=\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

76. If three concurrent lines from the angles of a triangle $\triangle ABC$ meet the opposite side in the points A', B', C', and the points A', B', C' are joined, forming a second triangle $\triangle A'B'C'$, then

$$\triangle ABC : \triangle A'B'C' = \overline{AB} \cdot \overline{BC} \cdot \overline{CA} : 2 \cdot \overline{AB'} \cdot \overline{BC'} \cdot \overline{CA'}$$

77. In the same case as #76, find the diameter of the circle circumscribed about the triangle $\triangle ABC = \overline{AB'} \cdot \overline{BC'} \cdot \overline{CA'}$ divided by the area of $A'B'C'$.

78. If a quadrilateral is inscribed in one circle and circumscribed to another, the square of its area is equal to the product of its four sides.

79. If on the sides $\overline{AB}$, $\overline{AC}$ of a triangle $\triangle ABC$ we take two points D, E on their connecting segment such that

$$\frac{BD}{AD} = \frac{AE}{CE} = \frac{DE}{EF}$$

then prove that $\triangle BFC = 2 \cdot \triangle ADE$.

80. If through the midpoints of each of the two diagonals of a quadrilateral we construct a parallel to the other, the lines constructed from their points of intersection to the midpoints of the sides divide the quadrilateral into four equal parts.

81. Suppose that $\overline{CE}$, $\overline{DF}$ are perpendiculars to the diameter of a semicircle, and two circles are constructed touching $\overline{CE}$, $\overline{DE}$, and the semicircle, one internally and the other externally. Prove that the rectangle contained by the perpendiculars from their centers on $\overline{AB}$ is equal to $\overline{CE} \cdot \overline{DF}$.

82. If segments are constructed from any point in the circumference of a circle to the vertices of any inscribed regular polygon of an odd number of sides, the sums of the alternate lines are equal.

83. If at the endpoints of a chord constructed through a given point within a given circle tangents are constructed, the sum of the reciprocals of the perpendiculars from the point upon the tangents is constant.

84. If a cyclic quadrilateral is such that three of its sides pass through three fixed collinear points, the fourth side passes through a fourth fixed point, collinear with the three given ones.

85. If all the sides of a polygon are parallel to given lines and if the loci of all the angles except for one are segments, the locus of the remaining angle is also a segment.

86. If the vertical angle and the bisector of the vertical angle is given, the sum of the reciprocals of the containing sides is constant.

87. If P, P' denote the areas of two regular polygons of any common number of sides inscribed and circumscribed to a circle, and Π, Π' are the areas of the corresponding polygons of double the number of sides, prove that Π is a geometric mean between P and P' and Π' a harmonic mean between Π and P.

88. The difference of the areas of the triangles formed by joining the centers of the circles constructed about the equilateral triangles constructed outwards on the sides of any triangle is equal to the area of that triangle. Prove the same if they are constructed inwards.

89. In the same case as #88, the sum of the squares of the sides of the two new triangles is equal to the sum of the squares of the sides of the original triangle.

90. Suppose that R, r denote the radii of the circumscribed and inscribed circles to a regular polygon of any number of sides, R', r', corresponding radii to a regular polygon of the same area, and double the number of sides. Prove that $R' = \sqrt{Rr}$ and $r' = \sqrt{\frac{r(R+r)}{2}}$.

91. If the altitude of a triangle is equal to its base, the sum of the distances of the orthocenter from the base and from the midpoint of the base is equal to half the base.

92. In any triangle, when the base and the ratio of the sides are given, the radius of the circumscribed circle is to the radius of the circle which is the locus of the vertex as the difference of the squares of the sides is to four times the area.

93. Given the area of a parallelogram, one of its angles, and the difference between its diagonals, construct the parallelogram.

94. If a variable circle touches two equal circles, one internally and the other externally, and perpendiculars fall from its center on the transverse tangents to these circles, the rectangle of the intercepts between the feet of these perpendiculars and the intersection of the tangents is constant.

95. Given the base of a triangle, the vertical angle, and the point in the base whose distance from the vertex is equal half the sum of the sides, construct the triangle.

96. If the midpoint of the base BC of an isosceles triangle $\triangle ABC$ is the center of a circle touching the equal sides, prove that any variable tangent to the circle will cut the sides in points D, E, such that the rectangle $\overline{BD} \cdot \overline{CE}$ is constant.

97. Inscribe in a given circle a trapezoid, the sum of whose opposite parallel sides is given and whose area is given.

98. Inscribe in a given circle a polygon all of whose sides pass through given points.

99. If two circles $\bigcirc ABC$, $\bigcirc XYZ$ are related such that a triangle may be inscribed in $\bigcirc ABC$ and circumscribed about $\bigcirc XYZ$, prove that an infinite number of such triangles can be constructed.

100. In the same case as #99: the circle inscribed in the triangle formed by joining the points of contact on $\bigcirc XYZ$ touches a given circle.

101. In the same case as #99: the circle constructed about the triangle formed by drawing tangents to $\bigcirc ABC$ at the vertices of the inscribed triangle touches a given circle.

102. Find a point, the sum of whose distances from three given points is a minimum.

103. A line constructed through the intersection of two tangents to a circle is divided harmonically by the circle and the chord of contact.

104. Construct a quadrilateral similar to a given quadrilateral whose four sides pass through four given points.

105. Construct a quadrilateral similar to a given quadrilateral whose four vertices lie on four given lines.

106. Given the base of a triangle, the difference of the base angles, and the rectangle of the sides, construct the triangle.

107. Suppose that $\boxdot ABCD$ is a square, the side CD is bisected at E, and $\overline{EF}$ is constructed making the angle $\angle AEF = \angle EAB$. Prove that $\overline{EF}$ divides the side BC in the ratio of $2 : 1$.

108. If any chord is constructed through a fixed point on a diameter of a circle, its endpoints are joined to either end of the diameter, and the joining lines cut off on the tangent at the other end, then the portions whose rectangle is constant.

109. If two circles touch and through their point of intersection two secants be constructed at right angles to each other, cutting the circles respectively in the points A, A'; B, B'; then $(\overline{AA'})^2 + (\overline{BB'})^2$ is constant.

110. If two secants stand at right angles to each other which pass through one of the points of intersection of two circles also cut the circles again, and the line

through their centers is the two systems of points a, b, c; a', b', c' respectively, then $ab : bc = a'b' : b'c'$.

111. If a chord of a given circle stands opposite a right angle at a given point, the locus of the intersection of the tangents at its endpoints is a circle.

112. The rectangle contained by the segments of the base of a triangle made by the point of intersection of the inscribed circle is equal to the rectangle contained by the perpendiculars from the endpoints of the base on the bisector of the vertical angle.

113. If O is the center of the inscribed circle of the triangle, prove

$$\frac{OA^2}{bc} + \frac{OB^2}{ca} + \frac{OC^2}{ab} = 1$$

114. State and prove the corresponding theorems for the centers of the escribed circles.

115. Suppose that four points A, B, C, D are collinear. Find a point P at which the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ stand opposite equal angles.

116. The product of the bisectors of the three angles of a triangle whose sides are a, b, c, is

$$\frac{8abc \cdot s \cdot \text{area}}{(a+b)(b+c)(c+a)}$$

117. In the same case as #116, the product of the alternate segments of the sides made by the bisectors of the angles is

$$\frac{a^2b^2c^2}{(a+b)(b+c)(a+c)}$$

118. If three of the six points in which a circle meets the sides of any triangle are such that the lines joining them to the opposite vertices are concurrent, the same property is true of the three remaining points.

119. If a triangle $\triangle A'B'C'$ is inscribed in another $\triangle ABC$, prove

$$\overline{AB'} \cdot \overline{BC'} \cdot \overline{CA'} + \overline{A'B} \cdot \overline{B'C} \cdot \overline{C'A}$$

is equal to twice the triangle $\triangle A'B'C'$ multiplied by the diameter of the circle $\bigcirc ABC$.

120. Construct a polygon of an odd number of sides being given that the sides taken in order are divided in given ratios by fixed points.

121. If the external diagonal of a quadrilateral inscribed in a given circle is a chord of another given circle, the locus of its midpoint is a circle.

122. If a chord of one circle is a tangent to another, the line connecting the midpoint of each arc which it cuts off on the first to its point of intersection with the second passes through a given point.

123. From a point P in the plane of a given polygon, suppose that perpendiculars fall on its sides. If the area of the polygon formed by joining the feet of the perpendiculars is given, the locus of P is a circle.

124. The medians of a triangle divide each other in the ratio of $2 : 1$.

CHAPTER 7

Elementary Number Theory

7.1. Definitions

0. The Whole Numbers are a set of numbers which contains 0, 1, 2, 3, ... The positive integers are a subset of the Whole numbers, containing 1, 2, 3, ... Rational Numbers are the set of numbers that can be written $\frac{p}{q}$ where both p and q are whole numbers and $q \neq 0$. (Notice that the rational numbers include the whole numbers.) The Real Numbers consist of the rational numbers and the irrational numbers. We do not require the use of negative numbers.

1. We define the number 1 as the *unit number*, or more simply as the *unit*.

2. A number is certain multiple of units; or, if x is a number, then $x = xu = u + u + ... + u$ where $u = 1$ and u is added to itself a total of x times.

3. A number x is a *proper divisor* of a larger number y when there exists some whole number $n > 1$ such that $nx = y$. We may also write that x is a *proper factor* of y.

4. A number x is a *divisor* of a larger number y when there exists some whole number $n \geq 1$ where $nx = y$. We may also write that x is a *factor* of y.

5. If $x < y$ and there exists a whole number n such that $nx = y$, then y *is a multiple of* x.

6. An *even number* x is any whole number such that $x = 2b$ where b is whole number.

7. An *odd number* is any whole number which is not even. We may write such numbers as $x = 2b + 1$ where b is whole number.

8. An *even-times even number* x is any whole number such that $x = 4b$ where b is whole number.

9. An *even-times odd number* is any whole number which is the product of an even number and an odd number.

10. An *odd-times odd number* is any whole number which is the product of two odd numbers.

11. A *prime number* is any whole number greater than 1 which has only two proper divisors: 1 and itself.

12. Positive integers x and y are said to be *relatively prime* (or co-prime) if the only whole number that evenly divides both x and y is 1.

13. A *composite number* is any whole number which is not prime.

14. Positive integers x and y are said to be *relatively composite* if x and y are not relatively prime.

15. A whole number x is said to *multiply* a whole number y when $xy = y + y + ... + y$ where y is added to itself a total of x times.

16. When two numbers are multiplied to produce a third number, the third number is called a *product*. Since both numbers are positive, the product is positive, and it may be admissible to interpret the product as an area. (Euclid refers to these numbers as *plane numbers*.)

17. When three numbers are multiplied to produce a fourth number and the initial three numbers are positive, the product is positive, and it may be admissible to interpret the product as a volume. (Euclid refers to these numbers as *solid numbers*.)

18. A *square number* x is any number such that $x = b^2 = b \cdot b$ for some number b.

19. A *cube* (or a *cubed number*) x is any number such that $x = b^3 = b \cdot b \cdot b$ for some number b.

20. Four numbers a, b, c, and d are proportional whenever $a : b = c : d$ or equivalently when $\frac{a}{b} = \frac{c}{d}$.

21. A *perfect number* is that which is equal to the sum of its proper divisors/factors. One example is 6 since this number has 1, 2, and 3 as its factors and $6 = 1 + 2 + 3$.[1]

REMARK. Let k, x, and y be positive integers. The following statements are equivalent:

(1) $y = x \cdot k$

(2) x is a factor of y

(3) x is not relatively prime to y

(4) y is not relatively prime to x

[1]See `https://en.wikipedia.org/wiki/List_of_perfect_numbers` for a list of known perfect numbers.

(5) x divides y

(6) $x|y$

REMARK. Let x and y be positive integers. The following statements are equivalent:

(1) x is not a factor of y

(2) x is relatively prime to y

(3) y is relatively prime to x

(4) $x \nmid y$

7.2. Propositions from Book VII

PROPOSITION 7.1. *THE EUCLIDEAN ALGORITHM TO DETERMINE IF TWO NUMBERS ARE RELATIVELY PRIME.*

Let $x_1 > x_2$ be positive integers. Then $x_1 - n_1 \cdot x_2 > 0$ for some positive integer n_1 such that $x_1 - (n_1 + 1) \cdot x_2 < 0$; let $x_1 - n_1 \cdot x_2 = x_3$.

Repeat the algorithm k times: for $1, 2, ..., k$, choose n_k such that $x_k - n_k \cdot x_{k+1} > 0$ where $x_k - (n_k + 1) \cdot x_{k+1} < 0$. And if $x_k - (n_k + 1) \cdot x_{k+1} = x_{k+2}$, then x_{k+2} divides both x_k and x_{k+1}.

If there exists some index t such that $x_t - n_t \cdot x_{t+1} = 1$, then x_1 and x_2 are relatively prime.

PROOF. Let $x_1 > x_2$ be positive integers and apply the above algorithm. After a finite number of steps t, suppose we have that $x_t - n_t \cdot x_{t+1} = x_{t+2} = 1$. We claim that x_1 and x_2 are relatively prime.

Suppose otherwise: it follows that x_1 and x_2 share a common factor, and so there exists some positive integer $q > 1$ which divides both x_1 and x_2. Since $\frac{x_1}{q}$ and $\frac{x_2}{q}$ are positive integers, it follows that $\frac{x_1 - x_2}{q}$ is a positive integer (since $x_1 > x_2$).

This gives us the equality

$$\frac{x_1}{q} - n_1 \cdot \frac{x_2}{q} \quad = \quad \frac{x_3}{q}$$

where $\frac{x_3}{q}$ is a positive integer; in fact, each $\frac{x_k}{q}$ is a positive integer.

We repeat this algorithm until we obtain

$$\frac{x_t}{q} - n_t \cdot \frac{x_{t+1}}{q} \quad = \quad \frac{1}{q}$$

Hence $q = 1$ and $q > 1$, a contradiction. Thus, x_1 and x_2 are relatively prime. This proves our claim. □

EXAMPLE. of Proposition 7.1: are 363 and 19600 relatively prime?

Let $x_1 = 19600$ and $x_2 = 363$ since $x_1 > x_2$. Then

$$19600 - (53)(363) \quad = \quad 361$$

where $n_1 = 53$. Notice that

$$19600 - (54)(363) \quad = \quad -2$$

We continue:

$$363 - (1)(361) \quad = \quad 2$$

where $n_2 = 1$.

$$361 - (180)(2) \quad = \quad 1$$

where $n_3 = 180$, and so 363 and 19600 are relatively prime.

EXAMPLE. of Proposition 7.2: what is the greatest common factor (GCF) of 19600 and 42?

Let $x_1 = 19600$ and $x_2 = 42$ since $x_1 > x_2$. Then

$$\begin{aligned} 19600 - (466)(42) &= 28 \\ 42 - (1)(28) &= 14 \end{aligned}$$

Since 14 is a factor of 28, 14 is the GCF of 19600 and 42.

PROPOSITION 7.2. *THE GREATEST COMMON DIVISOR (aka. THE GREATEST COMMON FACTOR).*

Given two numbers which are not relatively prime, a *greatest common divisor* (GCD) exists (i.e., a largest number which divides both numbers).

We may also call this number the *greatest common factor* (GCF).

PROOF. Let $x_1 > x_2$ be numbers which are not relatively prime. We wish to find the GCD of x_1 and x_2.

If x_2 is a factor of x_1, then since x_2 is a factor of itself, x_2 is the GCD of x_1 and x_2 (because any number greater than x_2 cannot be a factor of x_2).

Now suppose that x_2 is not a factor of x_1. Using the algorithm in [7.1], after t steps we obtain

$$x_t - n_t \cdot x_{t+1} = x_{t+2}$$

where x_{t+2} is a factor of x_{t+1}; notice that $x_{t+2} \neq 1$ since x_1 and x_2 are not relatively prime.

Since x_{t+2} is a divisor of x_{t+1}, we also have that x_{t+2} is also a factor of x_1 and x_2.

Suppose k is a factor of x_1 and x_2. By the proof of [7.1], k is a factor of x_3, x_4, ..., and x_{t+2}. It follows that $k = x_{t+2}$.

Hence, x_{t+2} is the GCD is x_1 and x_2. □

COROLLARY. *7.2.1. If k divides both x and y, then k also divides the GCD of x and y.*

REMARK. David E. Joyce lists some of the assumptions Euclid makes about whole numbers at:

`http://aleph0.clarku.edu/~djoyce/java/elements/bookVII/propVII2.html`

Exercises.

1. Prove [Cor. 7.2.1].

PROPOSITION 7.3. *THE GCD OF THREE RELATIVELY COMPOSITE NUMBERS.*

PROOF. We claim a GCD exists for relatively composite numbers x_1, x_2, and x_3.

By [7.2], the GCD for x_1 and x_2 exists; denote it as y. Similarly, the GCD for y and x_3 exists; denote it as z. We claim that z is the GCD for x_1, x_2, and x_3.

Since $z|y$, $y|x_1$, and $y|x_2$, it follows that $z|x_1$ and $z|x_2$. Since $z|x_3$ by the above, we have proven our claim. □

EXAMPLE. of Proposition 7.3: what is the greatest common divisor (GCD) of 42, 282, and 900?

$$\begin{aligned} 282-(6)(42) &= 30 \\ 42-(1)(30) &= 12 \\ 30-(2)(12) &= 6 \end{aligned}$$

so $y = 6$. Then

$$900-(149)(6) = 6$$

and so $z = 6$. Hence, the GCD of 42, 282, and 900 is 6 (that is, if we factor 42, 282, and 900, the largest factor they will have in common is 6).

PROPOSITION 7.4. *FRACTIONS.*

Let $x_1 > x_2$ be positive integers. Then there exists a proper fraction $\frac{a}{b}$ such that $x_2 = \frac{a}{b} \cdot x_1$.

PROOF. Let $x_1 > x_2$ be positive integers. Notice that $0 < \frac{x_2}{x_1} < 1$, and so $\frac{x_2}{x_1}$ is a proper fraction.

Also notice that $x_2 = \frac{x_2}{x_1} \cdot x_1$. Write $\frac{x_2}{x_1} = \frac{a}{b}$ where $\frac{a}{b}$ is written in lowest terms. Then

$$x_2 = \frac{a}{b} \cdot x_1$$

Since $\frac{x_2}{x_1}$ is a proper fraction, $\frac{a}{b}$ is a proper fraction. This completes the proof. □

EXAMPLE. of Proposition 7.4: what is x when $6x = 47$?

[7.4] states that x is a proper fraction:

$$6 \cdot \frac{a}{b} \quad = \quad 47$$

Let $a = 47$ and $b = 6$; then $\frac{a}{b} = \frac{47}{6}$ is a proper fraction, and the equality holds.

EXAMPLE. of Proposition 7.4: what is x when $6x = 46$?

[7.4] states that x is a proper fraction:

$$6 \cdot \frac{a}{b} \quad = \quad 46$$

Let $a = 46$ and $b = 6$. This would give us $\frac{46}{6}$, which is not a proper fraction. But $\frac{46}{6} = \frac{23}{3}$ which is a proper fraction. So let $a = 23$, $b = 3$; then $\frac{a}{b} = \frac{23}{3}$ is a proper fraction, and the equality holds.

PROPOSITION 7.5. *DISTRIBUTION I.*

$$\frac{x}{b} + \frac{y}{b} = \frac{1}{b}(x + y)$$

PROOF. Let each x_i be a positive integer, and let $x_1 = \frac{x_2}{b}$ and $x_3 = \frac{x_4}{b}$ (by two applications of [7.4]). Then

$$\begin{aligned} x_1 + x_3 &= \frac{x_2}{b} + \frac{x_4}{b} \\ &= \frac{1}{b} \cdot x_2 + \frac{1}{b} \cdot x_4 \\ &= \frac{1}{b}(x_2 + x_4) \end{aligned}$$

□

COROLLARY. *7.5.1. [7.5] holds for any finite number of terms.*

PROPOSITION 7.6. *DISTRIBUTION II.*

$$\frac{a}{b} \cdot x + \frac{a}{b} \cdot y = \frac{a}{b}(x + y)$$

PROOF. Let each x_i be a positive integer, and let $x_1 = \frac{a}{b} \cdot x_2$ and $x_3 = \frac{a}{b} \cdot x_4$ (by two applications of [7.4]). Then

$$\begin{aligned} x_1 + x_3 &= \frac{a}{b} \cdot x_2 + \frac{a}{b} \cdot x_4 \\ &= \frac{a}{b}(x_2 + x_4) \end{aligned}$$

□

COROLLARY. *7.6.1. [7.6] holds for any finite number of terms.*

PROPOSITION 7.7. *DISTRIBUTION III.*

$$\frac{x_2}{b} - \frac{x_4}{b} = \frac{1}{b}(x_2 - x_4)$$

PROOF. Let each x_i be a positive integer, and let $x_1 = \frac{x_2}{b}$ and $x_3 = \frac{x_4}{b}$ (by two applications of [7.4]). Then

$$\begin{aligned} x_1 - x_3 &= \frac{x_2}{b} - \frac{x_4}{b} \\ &= \frac{1}{b} \cdot x_2 - \frac{1}{b} \cdot x_4 \\ &= \frac{1}{b}(x_2 - x_4) \end{aligned}$$

□

COROLLARY. *7.7.1. [7.7] holds for any finite number of terms.*

PROPOSITION 7.8. *DISTRIBUTION IV.*

$$\frac{a}{b} \cdot x - \frac{a}{b} \cdot y = \frac{a}{b}(x - y)$$

PROOF. Let each x_i be a positive integer, and let $x_1 = \frac{a}{b} \cdot x_2$ and $x_3 = \frac{a}{b} \cdot x_4$ (by two applications of [7.4]). Then

$$\begin{aligned} x_1 - x_3 &= \frac{a}{b} \cdot x_2 - \frac{a}{b} \cdot x_4 \\ &= \frac{a}{b}(x_2 - x_4) \end{aligned}$$

□

COROLLARY. *7.8.1. [7.8] holds for any finite number of terms.*

PROPOSITION 7.9. *MULTIPLICATION OF CERTAIN FRACTIONS I.*

PROOF. Let a, b, c, d, g, h, and n be positive integers such that $a = \frac{b}{n}$, $d = \frac{c}{n}$, and $a = \frac{g}{h} \cdot d$. It follows that

$$\begin{aligned} \frac{b}{n} &= \frac{g}{h} \cdot d \\ \frac{b}{n} &= \frac{g}{h} \cdot \frac{c}{n} \\ b &= \frac{g \cdot c}{h} \\ b &= \frac{g}{h} \cdot c \end{aligned}$$

□

PROPOSITION 7.10. *MULTIPLICATION OF CERTAIN FRACTIONS II.*

PROOF. Let a, b, c, d, g, h, and n be positive integers such that $a = \frac{b}{c} \cdot d$, $e = \frac{b}{c} \cdot f$, and $a = \frac{g}{h} \cdot e$. It follows that

$$\begin{aligned} \frac{b}{c} \cdot d &= \frac{g}{h} \cdot e \\ d &= \frac{c}{b} \cdot \frac{g}{h} \cdot e \\ d &= \frac{g}{h} \cdot \frac{c}{b} \cdot e \end{aligned}$$

Since $f = \frac{c}{b} \cdot e$, we have that

$$d = \frac{g}{h} \cdot f$$

□

PROPOSITION 7.11. *MANIPULATION OF RATIOS I.*

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a-c}{b-d} = \frac{c}{d}$$

PROOF. Let $a = \frac{n}{m} \cdot b$ where $n < m$, $c = \frac{r}{s} \cdot a$ where $r < s$, and $d = \frac{u}{v} \cdot b$ where $u < v$. Notice that $a > c$ and $b > d$.

Suppose that $a : b = c : d$. Then

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ \frac{n}{m} &= \frac{r}{s} \cdot \frac{n}{m} \cdot \frac{v}{u} \\ \frac{u}{v} &= \frac{r}{s} \end{aligned}$$

So $d = \frac{r}{s} \cdot b$. And so, $b - d = \frac{s-r}{s} \cdot b$ and $a - c = \frac{s-r}{s} \cdot a$, so

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ \frac{\frac{s-r}{s}}{\frac{s-r}{s}} \cdot \frac{a}{b} &= \frac{c}{d} \\ \frac{a-c}{b-d} &= \frac{c}{d} \end{aligned}$$

This completes the proof. □

EXAMPLE. of Proposition 7.11: since

$$\frac{4}{16} = \frac{1}{4}$$

then

$$\frac{4-1}{16-4} = \frac{3}{12} = \frac{1}{4}$$

PROPOSITION 7.12. *MANIPULATION OF RATIOS II.*

$$a_1 : b_1 = a_2 : b_2 = ... = a_n : b_n \Rightarrow \sum a_i : \sum b_i$$

PROOF. Suppose that $a_1 : b_1 = a_2 : b_2 = ... = a_n : b_n$.

Or, for $1 \leq i \leq n$,

$$\frac{a_i}{b_i} = k$$

It follows that

$$a_i = k \cdot b_i$$

for $1 \leq i \leq n$. Then

$$\begin{aligned} \sum a_i &= \sum k \cdot b_i \\ \sum a_i &= k \cdot \sum b_i \\ \frac{\sum a_i}{\sum b_i} &= k \end{aligned}$$

It follows that

$$\sum a_i : \sum b_i$$

which completes the proof. □

EXAMPLE. of Proposition 7.12: since

$$\frac{4}{16} = \frac{1}{4} = \frac{10}{40}$$

then

$$\frac{4+1+10}{16+4+40} = \frac{15}{60} = \frac{1}{4}$$

PROPOSITION 7.13. *MANIPULATION OF RATIOS III.*

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{c} = \frac{b}{d}$$

PROOF. Suppose $a : b = c : d$. Then

$$\begin{aligned} \frac{b}{c} \cdot \frac{a}{b} &= \frac{b}{c} \cdot \frac{c}{d} \\ \frac{a}{c} &= \frac{b}{d} \end{aligned}$$

or $a : c = b : d$. This completes the proof. □

EXAMPLE. of Proposition 7.13: since

$$\frac{4}{16} = \frac{1}{4}$$

then

$$\frac{4}{1} = \frac{16}{4}$$

PROPOSITION 7.14. *MANIPULATION OF RATIOS IV.*

Suppose that

$$\begin{aligned} a_1 : a_2 &= b_1 : b_2 \\ a_2 : a_3 &= b_2 : b_3 \\ ... &= ... \\ a_{n-1} : a_n &= b_{n-1} : b_n \end{aligned}$$

We claim that

$$a_1 : a_n = b_1 : b_n$$

PROOF. Our hypothesis and claim are stated above. Notice that for $1 \leq i \leq n-1$,

$$\frac{a_i}{a_{i+1}} = \frac{b_i}{b_{i+1}} \Rightarrow \frac{a_i}{b_i} = \frac{a_{i+1}}{b_{i+1}}$$

Then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_{n-1}}{b_{n-1}} = \frac{a_n}{b_n} \Rightarrow \frac{a_1}{a_n} = \frac{b_1}{b_n}$$

which is equivalent to

$$a_1 : a_n = b_1 : b_n$$

This proves our claim. □

EXAMPLE. of Proposition 7.12: since

$$\frac{4}{16} = \frac{1}{4}, \text{ and } \frac{16}{160} = \frac{4}{40}$$

then

$$\frac{4}{160} = \frac{1}{40}$$

PROPOSITION 7.15. *COMMUTATIVITY OF MULTIPLICATION, part 1.*

Suppose that $b = nu$, $e = nd$, and $d = mu$ where $u = 1$. We claim that $e = mb$.

PROOF. Our hypothesis and claim are stated above.[2]

Recall that [7.12] states that if $a_1 : b_1 = a_2 : b_2 = ... = a_n : b_n$, then $(a_1 + a_2 + ... + a_n) : (b_1 + b_2 + ... + b_n)$. Let each $a_i = 1$ and each $b_i = d$. It follows that

$$\frac{u}{d} = \frac{nu}{nd}$$

Since $d = mu$ by hypothesis, we have

$$\frac{u}{mu} = \frac{nu}{n(mu)}$$

or $un(mu) = mu(nu)$. Since $u = 1$, we may obtain $n(mu) = m(nu)$.

Since

$$\begin{aligned} n(mu) &= m(nu) \\ nd &= mb \\ e &= mb \end{aligned}$$

our claim is proven. □

PROPOSITION 7.16. *COMMUTATIVITY OF MULTIPLICATION, part 2.*

$$ab = ba$$

PROOF. We claim that if a and b are real numbers then $ab = ba$.

Let $a = nu$ and $b = mu$ where $u = 1$. By the definition of multiplication, we have $ab = nb = n(mu)$ and $ba = ma = m(nu)$.

By the proof of [7.15], $n(mu) = m(nu)$. This proves our claim. □

[2]The proofs to [7.15] and [7.16] are based on David E. Joyce's proofs.
http://aleph0.clarku.edu/~djoyce/java/elements/bookVII/propVII15.html
http://aleph0.clarku.edu/~djoyce/java/elements/bookVII/propVII15.html

PROPOSITION 7.17. *MULTIPLYING BY ONE I.*

$$\frac{a}{b} = \frac{ka}{kb}$$

PROOF. Suppose $a : b$. It follows that

$$\begin{aligned}\frac{a}{b} &= \frac{a}{b} \\ &= 1 \cdot \frac{a}{b} \\ &= \frac{k}{k} \cdot \frac{a}{b} \\ &= \frac{ka}{kb}\end{aligned}$$

□

PROPOSITION 7.18. *MULTIPLYING BY ONE II.*

$$\frac{a}{b} = \frac{ak}{bk}$$

PROOF. The proof follows immediately from [7.17] and [7.16]. □

PROPOSITION 7.19. *EQUIVALENCE OF PROPORTIONS.*

$$a : b \quad = \quad c : d \text{ iff } ad = bc$$

PROOF. Suppose $a : b = c : d$ for positive real numbers a, b, c, and d; equivalently, suppose $\frac{a}{b} = \frac{c}{d}$. We claim that $ad = bc$.

Notice that

$$\begin{aligned} \frac{a}{b} \cdot \frac{db}{1} &= \frac{c}{d} \cdot \frac{db}{1} \\ ad &= cb \\ ad &= bc \end{aligned}$$

Now suppose that $ad = bc$; we claim that $\frac{a}{b} = \frac{c}{d}$.

$$\begin{aligned} \frac{ad}{bd} &= \frac{bc}{bd} \\ \frac{a}{b} &= \frac{c}{d} \end{aligned}$$

This proves our claim. □

PROPOSITION 7.20. *REDUCTION TO LOWEST TERMS.*

If $a : b$ reduces to $c : e$ in lowest terms, then c divides (is a factor of) a.

PROOF. Suppose that

$$\frac{a}{b} = \frac{c}{e}$$

where $a : b$ reduces to $c : e$ in lowest terms. We claim that c is a factor of a.

Suppose instead that $c = \frac{x}{y} \cdot a$, and $e = \frac{x}{y} \cdot b$. Since $\frac{c}{x} = \frac{a}{y}$ and $\frac{e}{x} = \frac{b}{y}$,

$$\frac{c}{x} : \frac{e}{x} = a : b$$

and so $c : e$ is not in lowest terms, a contradiction. Therefore, $c|a$ (or c is a factor of a), which proves our claim. □

COROLLARY. *7.20.1 If $a : b$ reduces to $c : e$ in lowest terms, then e divides b.*

PROPOSITION 7.21. *RELATIVE PRIMES I.*

If a is relatively prime to b, then the ratio $a : b$ is reduced to lowest terms.

PROOF. Suppose a is relatively prime to b. We claim that the ratio $a : b$ is reduced to lowest terms.

Since a and b are relatively prime, any list of prime factors of a will not contain any prime factors of b.

But suppose that $a : b$ is not reduced to lowest terms: then there exists some prime factor n such that $nk = a$ and $nm = b$ where $k : m$ is $a : b$ written in lowest terms (where k, m, and n are positive integers).

This contradicts the fact that a and b share no prime factors, which proves our claim. □

PROPOSITION 7.22. *RELATIVE PRIMES II.*

If the ratio $a : b$ is reduced to lowest terms, then a is relatively prime to b.

PROOF. Suppose the ratio $a : b$ is reduced to lowest terms. We claim that a is relatively prime to b.

Suppose instead that a is not relatively prime to b. Then there exists some prime factor n that both a and b share. Then there exist some x and y such that $nx = a$ and $ny = b$ (where n, x, and y are positive integers). Thus $a : b$ written in lowest terms is $x : y$, a contradiction.

This proves our claim. □

COROLLARY. *7.22.1 [7.21] and [7.22] state that the ratio $a : b$ is written in lowest terms iff a is relatively prime to b.*

PROPOSITION 7.23. *RELATIVE PRIMES III.*

If a and b are relatively prime and $m|a$, then b and m are also relatively prime.

PROOF. Our hypothesis and claim are stated above.

Assume instead there exists a positive integer n such that $n|m$ and $n|b$. Since $n|m$ and $m|a$, $n|a$; this contradicts our assumption that a and b are relatively prime.

This proves our claim. □

PROPOSITION 7.24. *RELATIVE PRIMES IV.*

If x and y are relatively prime to z, then xy is also relatively prime to z.

PROOF. Our hypothesis and claim are stated above.

Since x is relatively prime to z, any list of prime factors of x will not contain any prime factors of z. Similarly, any list of prime factors of y will not contain any prime factors of z.

Since any list of prime factors of xy can only contain prime factors of x and y, such a list will not contain any prime factors of z, which proves our claim. □

PROPOSITION 7.25. *RELATIVE PRIMES V.*

If x and y are relatively prime, then x^2 is relatively prime to y.

PROOF. Our hypothesis and claim are stated above. In [7.24], let $y = x$ and $z = y$; the claim is proven immediately. □

PROPOSITION 7.26. *RELATIVE PRIMES VI.*

If x is relatively prime to s and t, and if y is also relatively prime to s and t, then xy is relatively prime to st.

PROOF. Our hypothesis and claim are stated above.

Since x and y are each relatively prime to s, xy is also relatively prime to s by [7.24]. Similarly, xy is relatively prime to t.

Since s and t are each relatively prime to xy, st is relatively prime to xy by a final application of [7.24]. This proves our claim. □

PROPOSITION 7.27. *RELATIVE PRIMES VII.*

If x and y are relatively prime, then x^2 and y^2 are relatively prime.

PROOF. Our hypothesis and claim are stated above.

Since x and y are relatively prime, by [7.25] x^2 and y are relatively prime; and since y and x^2 are relatively prime, by a second application of [7.25] y^2 and x^2 are relatively prime. This proves our claim. □

REMARK. Euclid also uses [7.27] to prove that if x and y are relatively prime, then x^3 and y^3 are relatively prime. We prove this is [Cor. 7.27.1].

COROLLARY. *7.27.1. If x and y are relatively prime, then x^k and y^k are relatively prime where $k > 1$ is any positive integer.*

PROOF. A proof by induction[3]: by [7.27], if x and y are relatively prime, then x^2 and y^2 are relatively prime.

Suppose that if x and y are relatively prime, then x^n and y^n are relatively prime where $n \geq 2$ is any positive integer. We wish to show that x^{n+1} and y^{n+1} are relatively prime.

[3]See `http://www.people.vcu.edu/~rhammack/BookOfProof/Induction.pdf` and `http://www.basic-mathematics.com/proof-by-mathematical-induction.html`

Let the prime factorization of $x = u_1 \cdot u_2 \cdot ... \cdot u_k$, and let the prime factorization of $y = v_1 \cdot v_2 \cdot ... \cdot v_g$. Since x and y are relatively prime, $u_i \neq v_j$ for $1 \leq i \leq k$ and $1 \leq j \leq g$.

Then $x^n = u_1^n \cdot u_2^n \cdot ...u_k^n$ and $y^n = v_1^n \cdot v_2^n \cdot ...v_g^n$. By the above, we have that $u_i^n \neq v_j^n$ for $1 \leq i \leq k$ and $1 \leq j \leq g$. Hence, $u_i^{n+1} \neq v_j^{n+1}$ for $1 \leq i \leq k$ and $1 \leq j \leq g$. But this implies that x^{n+1} and y^{n+1} are relatively prime, which proves our claim. □

PROPOSITION 7.28. *RELATIVE PRIMES VIII.*

(1) If x and y are relatively prime, then $x + y$ is relatively prime to both x and y.

(2) If $x + y$ is relatively prime to either x or to y, then x and y are relatively prime.

PROOF. We prove each claim separately.

Claim 1: if x and y are relatively prime, then $x + y$ is relatively prime to both x and y.

Assume instead there exists some positive integer n such that $\frac{x+y}{n}$ and $\frac{x}{n}$ are integers. Notice that $\frac{x+y}{n}$ can be written $\frac{x}{n} + \frac{y}{n}$, and so $n|x$ and $n|y$. But x and y are relatively prime, a contradiction.

Therefore, $x + y$ is relatively prime to x. Similarly, $x + y$ is relatively prime to y. This proves claim 1.

Claim 2: if $x + y$ is relatively prime to x, then x and y are relatively prime.

Assume otherwise: then there exists some positive integer n such that $\frac{x}{n}$ and $\frac{y}{n}$ are integers.

By claim 1, $\frac{x+y}{n}$ is an integer, so $n|x$ and $n|x+y$. This contradicts our hypothesis that $x+y$ is relatively prime to x. A similar result if obtained if we suppose that $x+y$ is relatively prime to y. This proves claim 2, and the proof is complete. □

REMARK. Notice that [7.28] cannot be written using an "if and only if" statement without discarding part of the result.

PROPOSITION 7.29. *PRIME NUMBERS I.*

A prime number is relatively prime to any number of which it is not a factor.

PROOF. Suppose x is a prime number and y is any number which does not have x as a factor. We claim that x and y are relatively prime.

Assume instead there exists some positive integer $n > 1$ such that $n|x$ and $n|y$.

Since n is a factor of y and x is not a factor of y, $n \neq x$. But since $n|x$ where x is a prime number, $n = x$, a contradiction.

This proves our claim. □

PROPOSITION 7.30. *PRIME NUMBERS II.*

Suppose x and y are positive integers and p is a prime number. If p is a factor of xy, then p is a factor of either x or y.

PROOF. Our hypothesis and claim are stated above.

Suppose that p is not a factor of x; hence, p and x are relatively prime [7.29], and there exists some positive integer $n > 1$ such that $np = xy$. It follows that $\frac{p}{x} = \frac{y}{n}$ [7.19].

By [7.21], since p and x are relatively prime, $\frac{p}{x}$ is written in lowest terms. And by [7.20], p is a factor of y.

Similarly, we can show that if p is not a factor of y, then p is a factor of x. This proves our claim. □

LEMMA. *7.30.1 If S is a set of strictly decreasing positive integers which contains a maximum element, then S is a finite set.*

PROPOSITION 7.31. *PRIME NUMBERS III.*

Let x be any composite number. There exists some prime number p such that p is a factor of x.

PROOF. Our hypothesis and claim are stated above.

Since x is composite, there exists some number y which is a factor of x. If y is prime, then our claim is proven. Otherwise, suppose z is a factor of y.

Clearly, z is also a factor of x. If z is prime, then our claim is proven. Otherwise, suppose z' is a factor of z.

We may continue this algorithm as many times as necessary. Suppose we do not find some number p which is prime and a factor of z' (and hence of x). It follows that we have constructed an infinite set of strictly decreasing factors of x whose largest element is x itself; this contradicts Lemma 7.30.1.

Hence, we eventually find some prime number p which is a factor of x. □

PROPOSITION 7.32. *PRIME NUMBERS IV.*

Any positive integer is either a prime number or has a prime number as a factor.

PROOF. Let x be any positive integer; we claim that x is either prime or has a prime number as a factor.

If x is prime, then our claim is proven.

Otherwise, x is composite; by [7.31], there exists a prime number p such that p is a factor of x. The proof follows. □

PROPOSITION 7.33. *FINDING RATIOS IN LOWEST TERMS.*

Given positive integers a, b, c and the ratio $a : b : c$, we may find the ratio $d : e : f$ such that $a : b : c = d : e : f$ where no smaller integers exist to make this equality true.

PROOF. Our hypothesis and claim are stated above.

If a, b, and c are relatively prime, then let $a = d$, $b = e$, and $c = f$, and our claim is proven by [7.21].

Otherwise, by [7.3] a, b, and c have a greatest common factor g. Let $gd = a$, $ge = b$, and $gf = c$. Hence, $a : b : c = d : e : f$.

We now wish to show that no smaller positive integers exist which make this equality true.

Suppose d, e, and f are not the least such positive integers; instead, suppose h, j, and k are respectively less than d, e, and f such that $a : b : c = h : j : k$. Hence there exists positive integer m such that $mh = a$, $mj = b$, and $mk = c$.

Suppose $m = g$; then $md = gd = a = mh$, and so $h = d$. But $h < d$ by hypothesis, a contradiction.

Now suppose $m < g$; since $mh = a = gd$, it follows that $h > d$; since $h < d$ by hypothesis, we have another contradiction.

Finally, suppose $m > g$; then g is not the greatest common factor of a, b, and c, a contradiction.

This proves our claim. □

REMARK. Euclid also uses [7.33] to prove that if n positive integers are respectively proportional, then we may write such proportions in lowest terms. We state this is [Cor. 7.33.1].

COROLLARY. *7.33.1. Given positive integers x_1, x_2, ..., x_n and the ratio $x_1 : x_2 : ... : x_n$, we may find the ratio $y_1 : y_2 : ... : y_n$ such that $x_1 : x_2 : ... : x_n = y_1 : y_2 : ... : y_n$ such that no smaller integers than each y_i exist which make this equality true.*

Exercises.

1. Prove [Cor. 7.33.1].

PROPOSITION 7.34. *LEAST COMMON MULTIPLE I.*

Given x and y, we claim that a smallest positive integer exists which has x and y as factors. This is number is the *Least Common Multiple* (LCM) of x and y.

PROOF. Our hypothesis and claim are stated above.

Either x and y are relatively prime or they are not.

Case 1: x and y are relatively prime. Since x and y each divide xy, we claim that xy is the smallest positive integer which has x and y as a factor.

If not, suppose that x and y divide some number $d < xy$, and there exist positive integers $e > 1$ and $f > 1$ such that $ex = d$ and $fy = d$. It follows that $ex = fy$ and $\frac{x}{y} = \frac{f}{e}$.

Since x and y are relatively prime, $\frac{x}{y}$ is written in lowest terms by [7.21]; since $\frac{x}{y} = \frac{f}{e}$, by [7.20] x is a factor of f, and by [7.20.1] y is a factor of e. And so there exist positive integers $m > 1$ and $n > 1$ such that $mx = f$ and $ny = e$; thus, $mxy = d$ and $nxy = d$. It follows that $m = n$. Then

$$xy = \frac{d}{m} > d$$

a contradiction.

Therefore, xy is the least common multiple of x and y.

Case 2: x and y are not relatively prime. Let $e > 1$ and $f > 1$ be positive integers such that $\frac{x}{y} = \frac{f}{e}$ where $\frac{f}{e}$ is written in lowest terms. It follows that $xe = yf$.

Let $xe = yf = z$. We claim that z is the the smallest positive integer which has x and y as a factor.

Otherwise, x and y divide some $d < z$. Then there exists some positive integer $g > 1$ such that $xg = d$ and some positive integer $h > 1$ such that $yh = d$. It follows that $xg = yh$, and so $\frac{x}{y} = \frac{h}{g}$.

By the above, we have that $\frac{f}{e} = \frac{h}{g}$. Since $\frac{f}{e}$ are written in least terms of all fractions equaling $\frac{x}{y}$, e divides g [7.20.1].

Since $xe = z$ and $xg = d$, it follows that $\frac{z}{e} = \frac{d}{g}$, and $\frac{z}{d} = \frac{e}{g}$. Since $e|g$, we must have that $z|d$. Thus $z < d$ and $d < z$, a contradiction.

Hence, z is the least common multiple of x and y. □

PROPOSITION 7.35. *LEAST COMMON MULTIPLE II.*

If x and y divide z, and if a is the least common multiple of x and y, then a also divides z.

PROOF. Our hypothesis and claim are stated above.[4]

Suppose a does not divide z. Let $z = nk + f$ where n and k are positive integers, a is a factor of nk, and $f < a$.

Since x and y divide a, and a divides nk, it follows that x and y also divide nk.

But x and y also divide z; since $f = z - nk$, x and y also divide f. Since $f < a$ by hypothesis, we have found a common multiple of x and y less than a, a contradiction.

This proves our claim. □

EXAMPLE. of Proposition 7.34: consider 6 and 14. Numbers which have 6 and 14 as factors include 42, 82, 168, and so on. However, 42 is the smallest such number, making it the LCM of 6 and 14.

EXAMPLE. of Proposition 7.35: 6 and 14 are each factors of 168. Therefore, 42 is also a factor of 168.

[4]The proof of [7.35] follows David E. Joyce's outline of Euclid's proof.
http://aleph0.clarku.edu/~djoyce/java/elements/bookVII/propVII35.html

PROPOSITION 7.36. *LEAST COMMON MULTIPLE III.*

The least common multiple of x, y, and z exists whenever x, y, and z are positive integers.

PROOF. Let x, y, and z be positive integers. We wish to find their least common multiple.

Let d be the least common multiple of x and y. Either z divides d nor not.

Claim 1: if $z|d$, then d is the least common multiple of x, y, and z.

Suppose instead that x, y, and z divide $e < d$ where e is a positive integer. By [7.35], the least common multiple of x and y divides e. But d is the least common multiple of x and y by hypothesis, and so $d|e$. Hence $d < e$ and $e < d$, a contradiction.

This proves our claim that d is the least common multiple of x, y, and z.

Claim 2: if $z \nmid d$, then the smallest positive integer such that both d and z are factors is the least common multiple of x, y, and z.

Let e be the smallest positive integer such that both d and z are factors.

Since x and y divide d, and d divides e, it follows that x and y also divide e. But z also divides e, and therefore e is a common multiple of x, y, and z.

If e is not the least common multiple of x, y, and z, then x, y, and z divide some positive integer $f < e$. Since x, y, and z divide f, it follows that x and y divide f; hence the least common multiple of x and y also divides f. Since d is the least common multiple divided by x and y, $d|f$.

But $z|f$, and so d and z divide f. Therefore the smallest positive integer divided by d and z also divides f.

Since e is the smallest positive integer divided by z and d, $e|f$; it follows that $e < f$ and $f < e$, a contradiction.

Hence, e is the least common multiple of x, y, and z. This proves our claim and completes the proof. □

COROLLARY. *7.36.1 The least common multiple of any finite set of numbers exists.*

PROPOSITION 7.37. *FACTORING I.*

Let x, y, and z be positive integers. If x is a factor of y, then there exists a positive integer z such that $y/x = z$.

PROOF. Our hypothesis and claim are stated above.

If y is prime, then either $x = 1$ and $z = y$, or $x = y$ and $z = 1$.

Otherwise, y is composite. Suppose $y = x_1 \cdot ... \cdot x_n \cdot z_1 \cdot ... \cdot z_n$. If $x = x_1 \cdot ... \cdot x_n$, then it follows that $z = z_1 \cdot ... \cdot z_n$; or $\frac{y}{x} = z$. □

PROPOSITION 7.38. *FACTORING II.*

Let x, y, and z be positive integers. If there exists a number z such that $y/x = z$, then x is a factor of y.

PROOF. Let x, y, and z be positive integers. Suppose that $\frac{y}{x} = z$. It follows that $y = xz$, and so x is a factor of y. □

COROLLARY. *7.38.1 Let x, y, and z be positive integers. [7.37] and [7.38] state that there exists a number z such that $y/x = z$ iff x is a factor of y.*

EXAMPLE. [Cor. 7.38.1] gives a number of results, including:

- Since $\frac{90}{9} = 10$, it follows that 9 is a factor of 90.
- Since 9 is a factor of 90, it follows that $\frac{90}{9}$ is a positive integer.
- Since 9 is not a factor of 10, it follows that $\frac{10}{9}$ is not a positive integer.
- Since $\frac{51}{6}$ is not a positive integer, 6 is not a factor of 51.

PROPOSITION 7.39. *FACTORING III.*

We wish to find the smallest positive integer a such that $\frac{a}{x}$, $\frac{a}{y}$, and $\frac{a}{z}$ are positive integers.

PROOF. Let a, x, y, and z be positive integers. We wish to find the smallest positive integer a such that $\frac{a}{x}$, $\frac{a}{y}$, and $\frac{a}{z}$ are positive integers.

Let g be the smallest positive integer that has x, y, and z as factors. It follows that $\frac{g}{x}$, $\frac{g}{y}$, and $\frac{g}{z}$ are positive integers. We claim that it is the smallest positive integer to do so.

Otherwise, there exists a positive integer $h < g$ such that $\frac{h}{x}$, $\frac{h}{y}$, and $\frac{h}{z}$ are positive integers. It follows that h has x, y, and z as factors. But g is the smallest positive integer that has x, y, and z as factors, and so $h > g$ and $h < g$, a contradiction.

Let $g = a$, and the proof is complete. □

CHAPTER 8

Proportions & Geometric Sequences

In this chapter, all variables a, b, c, etc., are assumed to be positive integers unless otherwise noted.

8.1. Definitions

1. A *continued proportion* is equivalent to a *geometric progression*. (Any theorem regarding one also applies to the other.) A continued proportion with a constant ratio will be an important topic in this chapter. One basic example is:

$$1 : 2 = 2 : 4 = 4 : 8 = 8 : 16$$

or

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16}$$

When simplified, each term equals $\frac{1}{2}$, but we refrain from simplifying the ratios because that would eliminate the continued proportion (which works as if it had links in a chain where opposing denominators and numerators are the links). Continued proportions may be said to be written in *lowest possible terms*. This is different from being written in *lowest terms*: the continued proportion above written in lowest terms is

$$1 : 2 = 1 : 2 = 1 : 2 = 1 : 2$$

which is no longer a continued proportion.

2a. Suppose $a = b \cdot c$, $d = e \cdot f$, and $b : c = e : f$ (or equivalently, $\frac{b}{c} = \frac{e}{f}$). We say that a and d are similar or write $a \sim d$. (Euclid describes a and b as *similar plane numbers*.)

2b. Similarly, if $g = h \cdot i \cdot j$, $k = l \cdot m \cdot n$, and $h : i : j = l : m : n$, we say that g and k are similar or write $g \sim k$. (Euclid describes g and k as *similar solid numbers*.) Notice that $h : i : j = l : m : n$ iff $h : l = i : m = j : n$.

3. When three numbers are continued proportions, the first is said to have to the third the *duplicate ratio* of that which it has to the second.

Algebraically,

$$\begin{aligned} \frac{x}{y} &= \frac{y}{z} \\ \left(\frac{x}{y}\right)^2 &= \frac{x}{y} \cdot \frac{y}{z} \\ \left(\frac{x}{y}\right)^2 &= \frac{x}{z} \end{aligned}$$

Example: $1 : 16$ *is the duplicate ratio of* $1 : 4$ *since*

$$\frac{1}{4} = \frac{4}{16}$$

and

$$\left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

4. When four numbers are continual proportionals, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second.

Algebraically,

$$\frac{u}{v} = \frac{v}{w} = \frac{w}{x}$$

so

$$\begin{aligned} \left(\frac{u}{v}\right)^3 &= \frac{u}{v} \cdot \frac{v}{w} \cdot \frac{w}{x} \\ \left(\frac{u}{v}\right)^3 &= \frac{u}{x} \end{aligned}$$

Example: $1 : 125$ *is the duplicate ratio of* $1 : 5$ *since*

$$\frac{1}{5} = \frac{5}{25} = \frac{25}{125}$$

and

$$\left(\frac{1}{5}\right)^3 = \frac{1}{125}$$

8.2. Propositions from Book VIII

PROPOSITION 8.1. *CONTINUED PROPORTIONS I.*

Suppose we have a continued proportion of positive integers such that

$$x_1 : x_2 = x_2 : x_3 = \ldots = x_{n-1} : x_n \tag{8.2.1}$$

and where x_1 and x_n are relatively prime, and where each fraction equals some rational number $\frac{s}{t} > 0$. Then for $2 \leq i \leq n$, each $x_{i-1} : x_i$ is written in lowest possible terms.

PROOF. Assume the above hypothesis. We claim that each $\frac{x_{i-1}}{x_i}$ for $2 \leq i \leq n$ is written in the lowest possible terms.

Assume otherwise: let

$$y_1 : y_2 = y_2 : y_3 = \ldots = y_{n-1} : y_n \tag{8.2.2}$$

be a set of ratios such that each term equals $\frac{s}{t}$ and where each term is written in lower terms than in (8.2.1). It follows that $y_1 < x_1$.

By [7.14], it follows that $\frac{x_1}{x_n} = \frac{y_1}{y_n}$. Recall that x_1 and x_n are relatively prime; by [7.21], $\frac{x_1}{x_n}$ is written in least terms. And by [7.20], x_1 is a factor of y_1, and so $y_1 > x_1$. But we also have that $y_1 < x_1$, a contradiction. Thus (8.2.1) is written in lower terms than (8.2.2).

Since (8.2.2) was chosen arbitrarily, (8.2.1) must be written in lowest possible terms. □

EXAMPLE. 8.1.1. Consider the continued proportion

$$16 : 40 = 40 : 100 = 100 : 250 = 250 : 625$$

16 and 625 are relatively prime, and hence each ratio is written in lowest possible terms. (If each ratio were written in lowest terms, each would equal $2 : 5$.)

PROPOSITION 8.2. *CONTINUED PROPORTIONS II.*

If $\frac{x}{y}$ is written in lowest terms, then there exists a continued proportion written in lowest possible terms where each term equals $\frac{x}{y}$.

PROOF. Assume the above hypothesis. We wish to find a continued proportion written in lowest possible terms where each term equals $\frac{x}{y}$.

Notice that for any positive integer $n \geq 2$ each term in the following continued proportion equals $\frac{x}{y}$:

$$\frac{x^n}{x^{n-1} \cdot y} = \frac{x^{n-1} \cdot y}{x^{n-2} \cdot y^2} = \ldots = \frac{x^2 \cdot y^{n-2}}{x \cdot y^{n-1}} = \frac{x \cdot y^{n-1}}{y^n} \tag{8.2.3}$$

We claim that all ratios in (8.2.3) are written in the lowest possible terms.

Since $\frac{x}{y}$ is written in least terms, by [Cor. 7.22.1] x and y are relatively prime. By [Cor. 7.27.1], x^n and y^n are relatively prime. And so by [8.1], each term in (8.2.3) is written in the lowest possible terms. Clearly this holds if $n = 2$, $n = k - 1$, and if $n = k$ for any positive integer $k > 2$.

Hence, the proof. □

REMARK. Euclid's original proof of [8.2] only covers the cases where $n = 2$ and $n = 3$.

EXAMPLE. 8.2.1. Consider the continued proportion

$$16 : 40 = 40 : 100 = 100 : 250 = 250 : 625$$

where $x = 2$, $y = 5$, and $n = 4$.

COROLLARY. *8.2.1 If we write the continued proportion (8.2.3) as a list of numbers, we obtain* a_1, a_2, ..., a_{n-1}, a_n. *Then* $a_1 = x^{n-1}$ *and* $a_n = y^{n-1}$ *for positive integers* x *and* y.

PROPOSITION 8.3. *CONTINUED PROPORTIONS III.*

Suppose we have a continued proportion of positive integers

$$y_1 : y_2 = y_2 : y_3 = \ldots = y_{n-1} : y_n$$

where each ratio is written in lowest possible terms. Then y_1 and y_n are relatively prime.

PROOF. Suppose we have a continued proportion of positive integers

$$\frac{x_1^n}{x_1^{n-1} \cdot x_2} = \frac{x_1^{n-1} \cdot x_2}{x_1^{n-2} \cdot x_2^2} = \ldots = \frac{x_1^2 \cdot x_2^{n-2}}{x_1 \cdot x_2^{n-1}} = \frac{x_1 \cdot x_2^{n-1}}{x_2^n} \tag{8.2.4}$$

where each ratio equals $\frac{x_1}{x_2}$ and is written in lowest possible terms ([8.2]). We claim that x_1^n and x_2^n are relatively prime.

By [7.33], a fraction in lowest terms exists that equals $\frac{x_1}{x_2}$; write this fraction as $\frac{a}{b}$. By [7.22], a and b are relatively prime; by [7.27], a^n and b^n are relatively prime. By [8.2], we have the continued proportion

$$\frac{a^n}{a^{n-1} \cdot b} = \frac{a^{n-1} \cdot b}{a^{n-2} \cdot b^2} = \ldots = \frac{a^2 \cdot b^{n-2}}{a \cdot b^{n-1}} = \frac{a \cdot b^{n-1}}{b^n} \tag{8.2.5}$$

such that each ratio equals $\frac{a}{b}$ and is written in lowest possible terms.

Since each ratio in (8.2.5) equals $\frac{a}{b}$ and is also written in lowest possible terms, then

$$\frac{x_1^{n-i} \cdot x_2^i}{x_1^{n-i-1} \cdot x_2^{i+1}} = \frac{a^{n-i} \cdot b^i}{a^{n-i-1} \cdot b^{i+1}}$$

for each $0 \leq i \leq n-1$. Since $a = x_1$ and $b = x_2$ and so x_1 and x_2 are relatively prime, x_1^n and x_2^n are relatively prime.

Let $y_j = x_1^{n+1-j} \cdot x_2^{j-1}$ for each $1 \leq j \leq n+1$; this proves our claim. □

EXAMPLE. 8.3.1. Consider the continued proportion

$$16 : 40 = 40 : 100 = 100 : 250 = 250 : 625$$

Each ratio is written in lowest possible terms, and so 16 and 625 are relatively prime.

COROLLARY. *8.3.1. Suppose we have a continued proportion of positive integers such that*

$$x_1 : x_2 = x_2 : x_3 = \ldots = x_{n-1} : x_n$$

By [8.1] and [8.3], we have that x_1 and x_n are relatively prime iff for $2 \leq i \leq n$ each $\frac{x_{i-1}}{x_i}$ is written in lowest possible terms.

PROPOSITION 8.4. *CONTINUED RATIOS I.*

Let $a : b$, $c : d$, and $e : f$ be written in lowest terms. There exists a continued ratio $x_1 : x_2 : x_3 : x_4$ written in lowest possible terms such that $\frac{x_1}{x_2} = \frac{a}{b}$, $\frac{x_2}{x_3} = \frac{c}{d}$, and $\frac{x_3}{x_4} = \frac{e}{f}$.

PROOF. Our hypothesis and claim are stated above.

Let $l_1 = mb = nc$, i.e., the least common multiple of b and c. We may write $\frac{a}{b} = \frac{ma}{mb}$ and $\frac{c}{d} = \frac{nc}{nd}$.

Let $l_2 = pnd = qe$, i.e., the least common multiple of nd and e. We may write $\frac{a}{b} = \frac{pma}{pmb}$, $\frac{c}{d} = \frac{pnc}{pnd}$, and $\frac{e}{f} = \frac{qe}{qf}$ where $pmb = pnc$ and $pnd = qe$. We claim that $pma : pmb : qe : qf$ is the required continued ratio.

Otherwise, there exists a continued ratio $x : y : z : u$ written in least possible terms exists such that $\frac{x_1}{x_2} = \frac{x}{y}$, $\frac{x_2}{x_3} = \frac{y}{z}$, and $\frac{x_3}{x_4} = \frac{z}{u}$. Notice that $l_2 \geq z$ since $x_3 = z \leq pnd = l_2$.

By [7.20], since $a : b$ is written in lowest terms and $a : b = x : y$, we have that b divides y. Similarly, since $c : d$ is written in lowest terms and $c : d = y : z$, we have that c divides y.

By [7.35], it follows that l_1 (the least common multiple of b and c) divides y. But

$$\frac{x_2}{x_3} = \frac{y}{z} = \frac{c}{d} = \frac{nc}{nd} = \frac{l_1}{nd}$$

and so nd divides z.

And since $\frac{e}{f} = \frac{z}{u}$ where $\frac{e}{f}$ is in lowest terms, e divides z; again by [7.35], l_2 (the least common multiple of nd and e) divides z. Hence $l_2 < z$ and $l_2 \geq z$, a contradiction.

This proves our claim. □

EXAMPLE. 8.4.1. Suppose we have $1 : 2$, $2 : 5$, and $3 : 4$. Then the required continued ratio is $3 : 6 : 15 : 20$. Equivalently, this can be written

$$3 : 6 = 6 : 15 = 15 : 20$$

or

$$\frac{3}{6} = \frac{6}{15} = \frac{15}{20}$$

PROPOSITION 8.5. *COMPOUND RATIOS I.*

If $a = cd$ and $b = ef$, then $\frac{a}{b} = \frac{c}{e} \cdot \frac{d}{f}$.

PROOF. Let $a = cd$ and let $b = ef$. Notice that

$$\frac{a}{b} = \frac{cd}{ef} = \frac{c}{e} \cdot \frac{d}{f}$$

□

PROPOSITION 8.6. *CONTINUED PROPORTIONS IV.*

Let $x_1 : x_2 = x_2 : x_3 = \ldots = x_{n-1} : x_n$ be a continued proportion. If x_1 does not divide x_2, then x_j does not divide x_k whenever $j < k \leq n$.

PROOF. Our hypothesis and claim are stated above.

Since x_1 does not divide x_2, they are relatively prime, and so $\frac{x_1}{x_2}$ is a positive fraction but not an integer. Hence each $\frac{x_{i-1}}{x_i}$ for any $3 \leq i \leq n$ is a positive fraction but not an integer; or, any $3 \leq i \leq n$, each x_i does not divide x_{i-1}.

Now suppose that there exists some $j < k$ such that x_j divides x_k.

Consider

$$\frac{x_{j-1}}{x_j} = \frac{x_{k-1}}{x_k}$$

It follows that

$$\frac{x_k}{x_j} = \frac{x_{k-1}}{x_{j-1}}$$

Since x_j divides x_k,

$$\frac{x_k}{x_j} = \frac{m}{1}$$

or $x_j = 1$. Then $x_k = 1$, and so $x_{j-1} = x_{k-1}$. It follows that

$$x_1 : x_2 = x_2 : x_3 = \ldots = x_{n-1} : x_n$$

is not a continued proportion, which contradicts our hypothesis.

This proves our claim. □

PROPOSITION 8.7. *CONTINUED PROPORTIONS V.*

Let $x_1 : x_2 = x_2 : x_3 = \ldots = x_{n-1} : x_n$ be a continued proportion. If x_1 divides x_n, then x_1 divides x_2.

PROOF. By [8.6], we have that for any continued proportion, if x_1 does not divide x_2, then x_1 does not divide x_n. The contrapositive of this statement is also true: if x_1 divides x_n, then x_1 divides x_2. This completes the proof. □

PROPOSITION 8.8. *CONTINUED PROPORTIONS VI.*

Let $a : b = e : f$ such that there exists a continued proportion $a : a_1 = a_1 : a_2 = \ldots = a_{n-1} : b$ (with $n - 1$ middle terms) where each term equals $a : b$. Then there exists a continued proportion $e : e_1 = e_1 : e_2 = \ldots = e_{n-1} : f$ where each term equals $e : f$ (also with $n - 1$ middle terms).

PROOF. Our hypothesis and claim are stated above.

Let

$$c : c_1 = c_1 : c_2 = ... = c_{n-1} : d$$

be written in the lowest possible terms such that each term is respectively equal to each term in $a : a_1 = a_1 : a_2 = \ldots = a_{n-1} : b$. By [8.3], c and d are relatively prime. By [7.21], $c : d$ is written in lowest terms.

Since $a : b = c : d$ and $a : b = e : f$, we have that $c : d = e : f$. By [7.20], it follows that $e = mc$, $f = md$ where $m > 0$ is an integer.

Then

$$mc : mc_1 = mc_1 : mc_2 = ... = mc_{n-1} : md$$

is a continued proportion where each term equals $mc : md = e : f$ with $n - 1$ middle terms. Let $mc_i = e_i$ and This proves our claim. □

REMARK. David E. Joyce points out that [8.8] can be used to prove that $\sqrt{2}$ is not a rational number.[1]

PROPOSITION 8.9. *CONTINUED PROPORTIONS VII.*

Let a and b be relatively prime, and suppose there exists n geometric means between a and b; then there exist n geometric means between 1 and a and between 1 and b.

PROOF. Our hypothesis and claim are stated above.

Suppose that a_1, a_2, ..., a_{n-2} are the geometric means between a and b; equivalently, suppose we have the continued proportion

$$a_0 : a_1 = a_1 : a_2 = \ldots = a_{n-2} : a_{n-1}$$

where $a_0 = a$ and $a_{n-1} = b$. Since a and b are relatively prime by hypothesis, by [8.1] each $a_{i-1} : a_i$ is written in lowest possible terms.

By the proof of [8.2], the terms of this continued proportion are respectively equal to the terms c^{n+1}, $c^n \cdot d$, $c^{n-1} \cdot d^2$, ..., $c \cdot d^n$, d^{n+1} where c and d are written in lowest terms [Cor 8.2.1].

Since, $a_0 = c^{n+1}$ and $a_{n-1} = d^{n+1}$, the continued proportions $1 : c = c : c^2 = c^2 : c^3 = \ldots = c^n : c^{n+1}$ and $1 : d = d : d^2 = d^2 : d^3 = \ldots = d^n : d^{n+1}$ exist with n geometric terms between 1 and c and between 1 and d. □

[1] http://aleph0.clarku.edu/~djoyce/java/elements/bookVIII/propVIII8.html

PROPOSITION 8.10. *CONTINUED PROPORTIONS VIII.*

Let there exist n geometric means between 1 and a and between 1 and b; then there exists n geometric means between a and b.

PROOF. Suppose the continued proportions $1 : c = c : c^2 = c^2 : c^3 = \ldots = c^n : c^{n+1}$ (defined as the first proportion) and $1 : d = d : d^2 = d^2 : d^3 = \ldots = d^n : d^{n+1}$ (defined as the second proportion) exist, each with n terms between 1 and c^{n+1} and between 1 and d^{n+1}. We claim that the continued proportion beginning with $a = c^{n+1}$ and ending with $b = d^{n+1}$ exists with n terms in-between.

Multiply term $n+1$ in the first proportion by term 1 in the second proportion. Then multiply term n in the first proportion by term 2 in the second proportion. Repeat this algorithm until we have a total of $n+1$ products. This gives us the terms:

$$c^{n+1},\ c^n \cdot d,\ c^{n-1} \cdot d^2,\ \ldots,\ c \cdot d^n,\ d^{n+1}$$

Clearly, we may write a continued proportion with these terms; notice that this proportion has n means between $a = c^{n+1}$ and $b = d^{n+1}$. □

REMARK. [2.10] is only a partial converse of [2.9]: [2.9] requires that a and b be relatively prime, but [2.10] does not.

PROPOSITION 8.11. *CONTINUED PROPORTIONS IX.*

Let a and b be positive integers. Then a^2 and b^2 have one and only one geometric mean (ab).

PROOF. Our hypothesis and claim are stated above. Notice that if

$$\frac{a^2}{x} = \frac{x}{b^2}$$

then

$$\begin{aligned} (ab)^2 &= x^2 \\ ab &= x \end{aligned}$$

since a, b, and x are positive integers. It follows that ab is a geometric mean of a^2 and b^2.

Suppose that c and d are positive integers such that $ab \neq cd$ and that cd is also a geometric mean of a^2 and b^2. Then

$$\begin{aligned} \frac{a^2}{cd} &= \frac{cd}{b^2} \\ (ab)^2 &= (cd)^2 \\ ab &= cd \end{aligned}$$

a contradiction. This proves our claim. □

PROPOSITION 8.12. *CONTINUED PROPORTIONS X.*

Let a and b be positive integers. Then a^3, a^2b, ab^2, and b^3 are in continued proportion.

PROOF. Assume the above hypothesis, and notice that

$$\frac{a}{b} = \frac{a^3}{a^2b} = \frac{a^2b}{ab^2} = \frac{ab^2}{b^3}$$

□

COROLLARY. *8.12.1 Let a^n and b^n be positive integers for any positive integer $n \geq 1$. Then there exist a set of geometric means to a^n and b^n such that*

$$\frac{a^n}{a^{n-1} \cdot b} = \frac{a^{n-1} \cdot b}{a^{n-2} \cdot b^2} = \ldots = \frac{a \cdot b^{n-1}}{b^n}$$

PROPOSITION 8.13. *CONTINUED PROPORTIONS XI.*

Suppose that $a_1 : a_2 = a_2 : a_3 = \ldots = a_{n-1} : a_n$. Then $a_1^k : a_2^k = a_2^k : a_3^k = \ldots = a_{n-1}^k : a_n^k$ for any positive integer $k \geq 1$.

PROOF. Our hypothesis and claim are stated above. Suppose we have a_i^k and a_j^k; then the terms

$$a_i^k,\ a_i^{k-1} \cdot a_j,\ a_i^{k-2} \cdot a_j^2,\ \ldots,\ a_i \cdot a_j^{k-1},\ a_j^k$$

are the geometric means between a_i^k and a_j^k. The common ratio of a continued proportion made of these terms is $a_i : a_j$.

Let $j = i + 1$. It follows that for $1 \leq i \leq n - 1$,

$$a_i^k : a_i^{k-1} \cdot a_{i+1} = a_i : a_{i+1}$$

Since each $a_i : a_{i+1}$ equals some term in

$$a_1 : a_2 = a_2 : a_3 = \ldots = a_{n-1} : a_n$$

then any pair of successive terms in $a_1 : a_2 = \ldots = a_{n-1} : a_n$ equals the same ratio as any pair of successive terms in $a_1^k : a_2^k = \ldots = a_{n-1}^k : a_n^k$. Since each continued proportion has $n + 1$ terms, our claim is proven. □

PROPOSITION 8.14. *FACTORS OF POWERS I.*

1. If a^2 is a factor of b^2, then a is a factor of b.
2. If a is a factor of b, then a^2 is a factor of b^2.

PROOF. Suppose a^2 is a factor of b^2. Then $b^2 = k^2 \cdot a^2$ where k is a positive real number. Hence, $b = ka$; or, a is a factor of b.

Now suppose that a is a factor of b. Then $b = k \cdot a$ where k is a positive real number, and

$$b^2 = (k \cdot a)^2 = k^2 \cdot a^2$$

Therefore, a^2 is a factor of b^2. □

REMARK. Euclid proves this hypothesis using [8.7]. However, it can be more easily solved via algebra.

PROPOSITION 8.15. *FACTORS OF POWERS II.*

1. If a^3 is a factor of b^3, then a is a factor of b.
2. If a is a factor of b, then a^3 is a factor of b^3.

PROOF. The proof follows the proof of [8.14], *mutatis mutandis*. □

COROLLARY. *8.15.1 If a^n is a factor of b^n for any positive integer n, then a is a factor of b. And if a is a factor of b, then a^n is a factor of b^n for any positive integer n.*

PROPOSITION 8.16. *FACTORS OF POWERS III.*

1. If a^2 is not a factor of b^2, then a is not a factor of b.
2. If a is not a factor of b, then a^2 is not a factor of b^2.

PROOF. Claim 1 is the contrapositive statement of [8.14], claim 2.

Claim 2 is the contrapositive statement of [8.14], claim 1. □

PROPOSITION 8.17. *FACTORS OF POWERS IV.*

1. If a^3 is not a factor of b^3, then a is not a factor of b.
2. If a is not a factor of b, then a^3 is not a factor of b^3.

PROOF. Claim 1 is the the contrapositive statement of [8.15], claim 2. Claim 2 is the contrapositive statement of [8.15], claim 1. □

COROLLARY. *8.17.1 If a^n is not a factor of b^n for any positive integer n, then a is not a factor of b. And if a is not a factor of b, then a^n is not a factor of b^n for any positive integer n.*

PROPOSITION 8.18. *GEOMETRIC MEANS I.*

If ab and cd are "similar plane numbers", i.e., products of factors a, b, c, and d such that $a : b = c : d$, then there exists one mean proportional between ab and cd. Also, ab is to cd in the duplicate ratio of a to c or of b to d.

PROOF. Assume the above hypothesis. By [7.13],

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{c} = \frac{b}{d}$$

Furthermore,

$$\frac{ab}{bc} = \frac{bc}{cd}$$

Hence, bc is a geometric mean between ab and cd. If x is also a geometric mean between ab and cd, then $x^2 = (bc)^2 = abcd$, and so $x = bc$; hence, bc is the geometric mean between ab and cd.

And since $\frac{a}{c} = \frac{b}{d}$,

$$\left(\frac{a}{c}\right)^2 = \frac{ab}{cd} = \left(\frac{b}{d}\right)^2$$

This completes the proof.

□

PROPOSITION 8.19. *GEOMETRIC MEANS II.*

If $a : b : c = d : e : f$, then there exist two geometric means between abc and def (specifically, bcd and cde). Also, abc is to def in the triplicate ratio of $a : d$, or $b : e$, or $c : f$.

PROOF. Our hypothesis and claim are stated above. Recall that $a : b : c = d : e : f$ iff $a : d = b : e = c : f$.

Notice that $a : d = abc : bcd$, that $b : e = bcd : cde$, and that $c : f = cde : def$. Hence,

$$\frac{abc}{bcd} = \frac{bcd}{cde} = \frac{cde}{def}$$

and so two geometric means, bcd and cde, exist between abc and def.

Since $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$,

$$\frac{a}{d} \cdot \frac{b}{e} \cdot \frac{c}{f} = \frac{abc}{def} = \left(\frac{a}{d}\right)^3 = \left(\frac{b}{e}\right)^3 = \left(\frac{c}{f}\right)^3$$

This completes the proof. □

PROPOSITION 8.20. *GEOMETRIC MEANS III.*

If a, c, and b are in geometrical progression, then a and b each have two factors and are similar.

PROOF. Suppose

$$\frac{a}{c} = \frac{c}{b}$$

We claim that a and b are similar and that each have two factors.

Let $\alpha : \gamma$ be the ratio $a : c$ reduced to lowest terms (as well as $c : b$ reduced to lowest terms). By [7.20], we have $a = m\alpha$, $c = m\gamma = n\alpha$, and $b = n\gamma$ where $m > 0$ and $n > 0$ are integers. Hence, a and b each have two factors.

Finally, notice that

$$\frac{\alpha}{\gamma} = \frac{a}{c} = \frac{c}{b} = \frac{m\gamma}{n\gamma} = \frac{m}{n}$$

By [7.13], $\alpha : m = \gamma : n$. By [Def VIII.2a], $a = m\alpha$ and $b = n\gamma$ are similar. □

PROPOSITION 8.21. *GEOMETRIC MEANS IV.*

If a, c, d, and b are in geometrical progression, then a and b each have three factors and are similar.

PROOF. Suppose

$$\frac{a}{c} = \frac{c}{d} = \frac{d}{b}$$

We claim that a and b each have three factors and are similar.

By either [7.33] or [8.2], let α, β, and γ be the least terms in the ratio of a, c, d (and therefore also of c, d, b). By [8.3], α and γ are relatively prime; by [8.20], α and γ are similar. Let $\alpha = mn$ and $\gamma = pq$ where $m : n = p : q$.

By the proof of [8.20],

$$\frac{\alpha}{\beta} = \frac{m}{p} = \frac{n}{q}$$

Since

$$\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{a}{c} = \frac{c}{d}$$

[7.14] states that $\frac{a}{d} = \frac{\alpha}{\gamma}$. And since α and γ are relatively prime, $a = r\alpha$ and $d = r\gamma$ where $r > 0$ is an integer. But $\alpha = mn$, so $a = rmn$.

Similarly, we can show that $\frac{c}{b} = \frac{\alpha}{\gamma}$, $c = s\alpha$, and $b = s\gamma$ where $s > 0$ is an integer. Since $\gamma = pq$, $b = spq$. It follows that a and b each have three factors.

And so

$$\alpha : \beta = a : c = r\alpha : s\alpha = r : s = m : p = n : q$$

The final three ratios show that $a = rmn$ and $b = spq$ are similar [Def VIII.2b]. □

PROPOSITION 8.22. *CONTINUED PROPORTIONS XII.*

If $\frac{a}{b} = \frac{b}{c}$ and $a = k^2$ for some integer $k > 0$, then $c = n^2$ for some integer $n > 0$.

PROOF. Assume the hypothesis, and notice that

$$\frac{k^2}{b} = \frac{b}{c} \Rightarrow\ k^2 c = b^2 \Rightarrow\ c = \left(\frac{b}{k}\right)^2$$

where $\frac{b}{k} = n$ is a positive integer. Therefore, $c = n^2$. □

PROPOSITION 8.23. *CONTINUED PROPORTIONS XIII.*

If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ and $a = n^3$ for some positive integer n, then $d = s^3$ for some integer $s > 0$.

PROOF. Our hypothesis and claim are stated above.

By [8.21], a and d are each the product of three factors, and a is similar to d. Let $a = nnn$ and $d = spq$. By the proof of [8.21], we have

$$n : s = n : p = n : q$$

For this equality to occur, we must have that $s = p = q$. It follows that $d = s^3$ for some integer $s > 0$. □

PROPOSITION 8.24. *CONTINUED PROPORTIONS XIV.*

If $a : b = c^2 : d^2$ and $a = s^2$ for some integer $s > 0$, then $b = t^2$ for some integer $t > 0$.

PROOF. Our hypothesis and claim are stated above.

Recall that $c^2 : cd = cd : d^2$ by [8.18]. And by [8.8], there exists one geometric mean between a and b. Since $a = s^2$, we have

$$\frac{s^2}{x} = \frac{x}{b} \Rightarrow b = \left(\frac{x}{s}\right)^2$$

where $\frac{x}{s}$ is a positive integer. Hence, $b = t^2$ for some integer $t > 0$. □

PROPOSITION 8.25. *CONTINUED PROPORTIONS XV.*

If $a : b = c^3 : d^3$ and $a = s^3$ for some integer $s > 0$, then $b = t^3$ for some integer $t > 0$.

PROOF. Our hypothesis and claim are stated above.

By [8.19], $c^3 : d^3$ have two mean proportionals. By [8.8], $a : b$ also have two mean proportionals. By [8.23], since $a = s^3$ for some integer $s > 0$, $b = t^3$ for some integer $t > 0$. □

PROPOSITION 8.26. *RATIOS OF SQUARE NUMBERS.*

Similar numbers which each have two factors have the same ratio to each other as a square number has to a square number.

PROOF. If a and b each have two factors and are similar, let c be the mean proportional between them [8.18].

Take α, β, and γ as the smallest terms in the ratio of a, c, and b (by either [7.33] or [8.2]).

By [Cor. 8.2.1], α and γ are square numbers. It follows that a and b have a ratio to each other as a square number has to a square number. □

COROLLARY. *8.26.1 If two numbers have a ratio to each other as a square number to a square number, then the numbers are similar plane numbers.*

PROPOSITION 8.27. *RATIOS OF CUBED NUMBERS.*

Similar numbers which each have three factors have a ratio to each other as a cubed number has to a cubed number.

PROOF. The proof to [8.27] is similar to the proof to [8.26] except that [8.19] is cited instead of [8.18]. □

COROLLARY. *8.27.1 If two numbers have to one another the ratio of a cubed number to a cubed number, the numbers are similar cubed numbers.*

Chapter 8 exercises.

1. Prove [Cor. 8.2.1].
2. Prove [Cor. 8.4.1].
3. Use [8.8] to prove that $\sqrt{2}$ is not a rational number.
4. Prove [Cor. 8.12.1].
5. Prove [Cor. 8.15.1].
6. Prove [Cor. 8.26.1].
7. Fill in the details for the proof of [8.27].
8. Prove [Cor. 8.27.1].

CHAPTER 9

Solutions

9.1. Solutions for Chapter 1

[1.1] Exercises

1. If the segments $\overline{AF}$ and $\overline{BF}$ are constructed, prove that the figure $\boxdot ACBF$ is a rhombus.

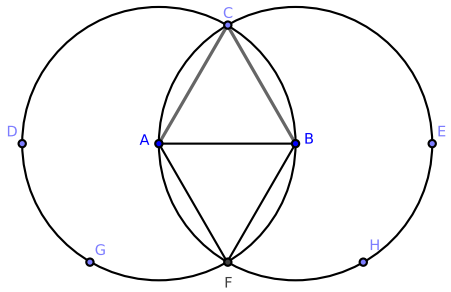


FIGURE 9.1.1. [1.1, #1]

PROOF. Construct $\overline{AF}$ and $\overline{BF}$. By an argument similar to the proof of [1.1], we have that $\overline{AB} = \overline{AF} = \overline{BF}$. Since we have $\overline{AC} = \overline{AB} = \overline{BC}$ from [1.1], if follows that

$$\overline{AC} = \overline{BC} = \overline{BF} = \overline{AF}$$

By [Def 1.29], $\boxdot ACBF$ is a rhombus. □

[1.1] Exercises

2. If $\overline{CF}$ is constructed and $\overline{AB}$ is extended to the circumferences of the circles (at points D and E), prove that the triangles $\triangle CDF$ and $\triangle CEF$ are equilateral.

PROOF. Construct $\overline{CF}$, and extend $\overline{AB}$ to $\overleftrightarrow{AB}$ where $\overleftrightarrow{AB}$ intersects $\bigcirc A$ at D and $\bigcirc B$ at E. Finally, construct $\triangle CDF$ and $\triangle CEF$. We wish to show that $\triangle CDF$ and $\triangle CEF$ are equilateral.

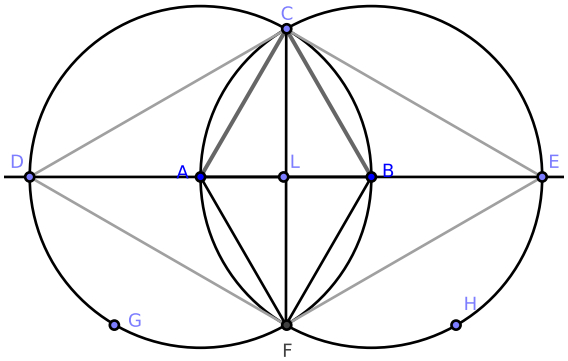


FIGURE 9.1.2. [1.1, #2]

Since $\boxdot ACBF$ is a rhombus by [1.1, #1], $\overline{CF}$ is its Axis of Symmetry. Consider $\triangle ACL$ and $\triangle BCL$. Since $\overline{CF}$ is the Axis of Symmetry of $\boxdot ACBF$, $\overline{AL} = \overline{LB}$. Since $\overline{AC} = \overline{BC}$ by [1.1] and $\triangle ACL$, $\triangle BCL$ share side LC, by [1.8], we have that $\triangle ACL \cong \triangle BCL$.

By [1.5, #5], $\overline{AB}$ is also an Axis of Symmetry of $\boxdot ACBF$. Consider $\triangle ACL$ and $\triangle AFL$. It follows that $\overline{CL} = \overline{LF}$. Since $\overline{AC} = \overline{AF}$ by [1.1] and $\triangle ACL$, $\triangle AFL$ share side AL, by [1.8], we have that $\triangle ACL \cong \triangle AFL$. Similarly, we can show that

$$\triangle ACL \cong \triangle AFL \cong \triangle BFL \cong \triangle BCL$$

The definition of a right angle states that it equals $\frac{\pi}{2}$ radians in measure. Notice that $\angle ALC = \frac{\pi}{2}$. By [1.32, #6], each angle of an equilateral triangle measures $\frac{\pi}{3}$ radians; hence $\angle LAC = \frac{\pi}{3}$. Since the interior angles of a triangle equal two right angles (π radians), it follows that $\angle LCA = \frac{\pi}{6}$. These results hold for the respective angles in all triangles congruent to $\triangle ACL$.

Notice that $\triangle ADC$ is an isosceles triangle since $AD = AC$; it follows that $\angle ADC = \angle ACD$ by [1.5]. Since $\angle LAC$ and $\angle DAC$ are supplemental angles, $\angle DAC = \frac{2\pi}{3}$ radians in measure. It follows that $\angle ADC = \angle ACD = \frac{\pi}{6}$.

Consider $\triangle DLC$: $\angle DLC = \frac{\pi}{2}$ and $\angle LCD = \frac{\pi}{3}$ since $\angle LCD = \angle ACL + \angle DCA$. It follows that $\angle LDC = \frac{\pi}{6}$. By similar reasoning, this holds for all respective angles in $\triangle ELC$, $\triangle ELF$, and $\triangle FLD$.

Consider $\triangle DLC$ and $\triangle ELC$: $DL = EL$, $\angle DLC = \angle ELC$, and they share side LC. By [1.4], $\triangle DLC \cong \triangle ELC$. A similar argument shows that $\triangle DLC \cong \triangle DLF$, from which it follows that

$$\triangle DLC \cong \triangle ELC \cong \triangle DLF \cong \triangle ELF$$

It follows that $\overline{CD} = \overline{EF} = \overline{FE} = \overline{EC}$.

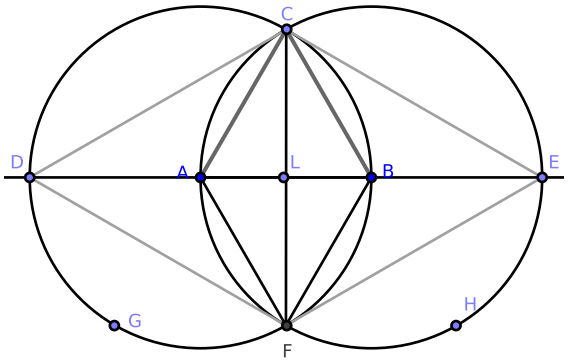


FIGURE 9.1.3. [1.1, #2]

Consider $\triangle DCF$ and $\triangle ECF$. By the above, $\angle FDC = \angle FEC = \frac{\pi}{3}$, $\angle DCF = \angle ECF = \frac{\pi}{3}$, and $\angle CFD = \angle CFE = \frac{\pi}{3}$; it follows that $\triangle DCF$ and $\triangle ECF$ are equilateral by [1.32, Cor. 6].

Hence, $\overline{CD} = \overline{EF} = \overline{FE} = \overline{EC} = \overline{CF}$; or, $\triangle CDF$ and $\triangle CEF$ are equilateral. □

COROLLARY. *1 from 1.1, #2.* $\boxdot CDFE$ *is a rhombus.*

[1.2] Exercises

1. Prove [1.2] when A is a point on $\overline{BC}$.

PROOF. Let $\overline{BC}$ be an arbitrary segment where A is a point on $\overline{BC}$. We claim that we can construct a segment with A as an endpoint such that its length is equal to that of $\overline{BC}$.

If $A = B$ or $A = C$, the proof follows trivially.

Suppose that A is not an endpoint of $\overline{BC}$. Construct the equilateral triangle $\triangle ABD$ [1.1]. Also construct the circle $\bigcirc A$ with radius equal in length to $\overline{AC}$. Extend side DA to $\overrightarrow{DA}$ where $\overrightarrow{DA}$ intersects $\bigcirc A$ at E.

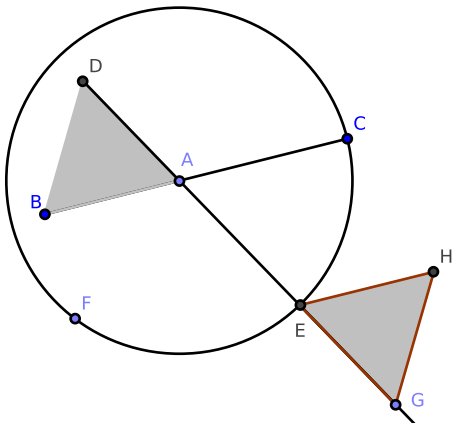


FIGURE 9.1.4. [1.2, #1]

Construct the equilateral triangle $\triangle EGH$ where G is also a point on $\overrightarrow{DA}$ and $\overline{EG} = \overline{AD}$ [1.2]. So $\overline{AE} \oplus \overline{EG} = \overline{AG}$ and

$$\begin{aligned} \overline{AG} &= \overline{AE} + \overline{EG} \\ &= \overline{AC} + \overline{AD} \\ &= \overline{AC} + \overline{AB} \\ &= \overline{BC} \end{aligned}$$

Hence, we have proven our claim. □

[1.4] Exercises

Prove the following:

1. The line that bisects the vertical angle of an isosceles triangle also bisects the base perpendicularly.

PROOF. Suppose that $\triangle ABC$ is an isosceles triangle where $AB = AC$. Further suppose that the ray AD bisects the angle $\angle BAC$. (A line, ray, or a segment of appropriate size may be used in this proof, *mutatis mutandis*.) We claim that $BD = CD$ and $AD \perp BC$.

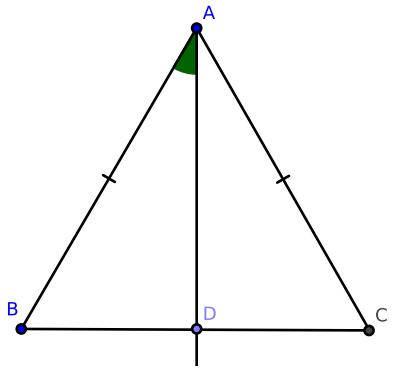


FIGURE 9.1.5. [1.4, #1]

Since $AB = AC$ and $\angle DAB = \angle DAC$ by hypothesis, and since $\triangle ABD$ and $\triangle ACD$ share the side AD, by [1.4] we have that

$$\triangle ABD \cong \triangle ACD$$

Hence, $BD = CD$ (claim 1) and $\angle ADB = \angle ADC$. Since the $\angle ADB$ and $\angle ADC$ are supplements, they are right angles by [Def. 1.14]; thus, $\overline{AD} \perp \overline{BC}$ (claim 2). □

[1.4] Exercises

2. If two adjacent sides of a quadrilateral are equal in length and the diagonal bisects the angle between them, then their remaining sides are also equal in length.

PROOF. Suppose that $ABCD$ is a quadrilateral where $AB = AC$ and where the diagonal AD bisects $\angle BAC$. We claim that $BD = CD$.

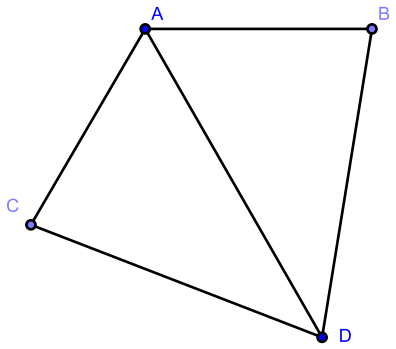


FIGURE 9.1.6. [1.4, #2]

Since $AC = AB$ and $\angle DAC = \angle DAB$ by hypothesis, and since $\triangle ACD$ and $\triangle ABD$ share the side AD, by [1.4] we have that $\triangle ACD \cong \triangle ABD$. Therefore, $BD = CD$. □

[1.4] Exercises

3. If two segments stand perpendicularly to each other and if each bisects the other, then any point on either segment is equally distant from the endpoints of the other segment.

PROOF. Suppose that $\overline{AB} \perp \overline{CD}$ and that $\overline{AB}$ and $\overline{CD}$ bisect each other at point E. Let F be a point on AB. We claim that any point on either segment is equally distant from the endpoints of the other segment.

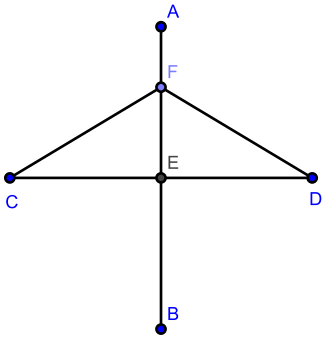


FIGURE 9.1.7. [1.4, #3]

Construct $\triangle CEF$ and $\triangle DEF$. Since $\overline{AB} \perp \overline{CD}$ and $\overline{AB}$ and $\overline{CD}$ bisect each other at point E by hypothesis, $\angle CEF = \angle DEF$ and $\overline{CE} = \overline{DE}$. Since $\triangle CEF$ and $\triangle DEF$ share side FE, by [1.4] we have that $\triangle CEF \cong \triangle DEF$ Hence, $CF = DF$.

The proof for any point on CD is similar to the above, *mutatis mutandis*. Therefore, we have proven our claim. □

COROLLARY. *to [1.4, #3]: Let $\overline{AB}$ and $\overline{CD}$ be two segments. If $\overline{AB} \perp \overline{CD}$, $\overline{CD}$ bisects $\overline{AB}$, and a triangle is formed on $\overline{AB}$ such that one vertex is a point on $\overline{CD}$, then $\overleftrightarrow{AB}$ is the Axis of Symmetry of the triangle.*

[1.5] Exercises

2. Prove that $\overleftrightarrow{AH}$ is an Axis of Symmetry of $\triangle ABC$.

PROOF. Construct the figure from [1.5], construct $\overleftrightarrow{AH}$, and let I be the intersection of $\overline{BC}$ and $\overleftrightarrow{AH}$. We claim that $\overleftrightarrow{AH}$ is the Axis of Symmetry of $\triangle ABC$.

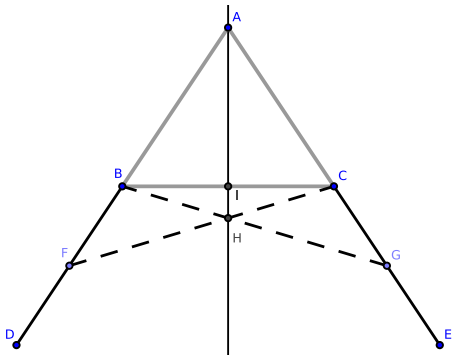


FIGURE 9.1.8. [1.5, #2]

Consider $\triangle BHF$ and $\triangle CHG$: by the proof of [1.5] $\angle BFH = \angle CGH$ and $BF = CG$, and by [1.15] $\angle BHF = \angle CHG$. By [1.26], $\triangle BHF \cong \triangle CHG$. It follows that $BH = CH$.

Consider $\triangle ABH$ and $\triangle ACH$: by the above, $BH = CH$; by the proof of [1.5] $AB = AC$; finally, the triangles share side AH. By [1.8], $\triangle ABH \cong \triangle ACH$. It follows that $\angle BAH = \angle CAH$.

Consider $\triangle ABI$ and $\triangle ACI$: by the above $\angle BAI = \angle CAI$, $AB = AC$, and the triangles share side AI. By [1.4], $\triangle ABI \cong \triangle ACI$. It follows that $\triangle ABI = \triangle ACI$.

Since $\triangle ABC = \triangle ABI \oplus \triangle ACI$, we have that $\overleftrightarrow{AH}$ is the Axis of Symmetry of $\triangle ABC$ by [Def 1.35]. This proves our claim. □

[1.5] Exercises

4. Take the midpoint on each side of an equilateral triangle; the segments joining them form a second equilateral triangle.

PROOF. Suppose that $\triangle ABC$ is equilateral. Construct the midpoints I, J, and K on sides BC, AB, and AC, respectively. We claim that $\triangle IJK$ is equilateral.

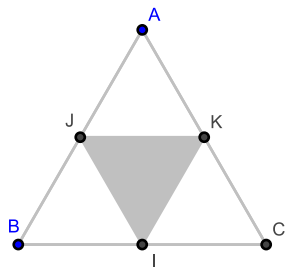


FIGURE 9.1.9. [1.5, #4]

Since I is the midpoint of side BC, $\overline{IB} = \overline{IC}$. Similarly, $\overline{JA} = \overline{JB}$. Since $\triangle ABC$ is equilateral, $\overline{IB} = \overline{JB}$. Continuing in this way, we can show that

$$\overline{IB} = \overline{JB} = \overline{JA} = \overline{AK} = \overline{KC} = \overline{IC}$$

And by [1.5, Cor. 1] we also have that

$$\angle ABC = \angle ACB = \angle BAC$$

Hence by multiple applications of [1.4],

$$\triangle JBI \cong \triangle KCI \cong \triangle JAK$$

It follows that $\overline{IJ} = \overline{JK} = \overline{KI}$, and so $\triangle IJK$ is equilateral. Our claim has been proved. □

[1.9] Exercises

2. Prove that $\overline{AF} \perp \overline{DE}$.

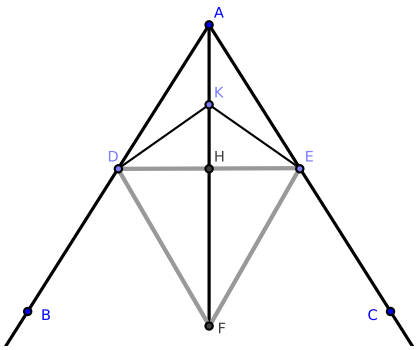


FIGURE 9.1.10. [1.9, #2]

PROOF. Since $\triangle ADE$ is isosceles, by [Cor. 1 from 1.5 #2], $\overline{AH} \perp \overline{DE}$. Since $\overline{AH}$ is a segment of $\overline{AF}$, the proof follows. □

3. Prove that any point on $\overline{AF}$ is equally distant from the points D and E.

PROOF. Let K be any point on $\overline{AF}$. We wish to show that $\overline{DK} = \overline{KE}$.

Consider $\triangle DHK$ and $\triangle EHK$: since $\triangle ADE$ is isosceles, by [Cor. 1 from 1.5 #2], $\overline{DH} = \overline{HE}$; the triangles share side $\overline{EH}$; since $AF \perp DE$, $\angle DHK = \angle EHK$. By [1.4], $\triangle DHK \cong \triangle EHK$. It follows that $\overline{DK} = \overline{KE}$, and our claim is proven. □

[1.10] Exercises

1. Bisect a segment by constructing two circles.

PROOF. Construct the figure from [1.1, #2]. We wish to bisect segment $\overline{AB}$.

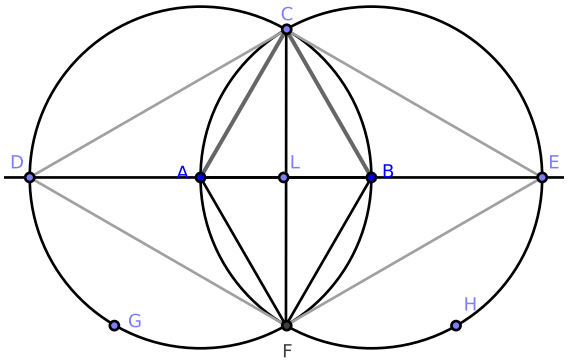


FIGURE 9.1.11. [1.10, #1]

The proof of [1.1, #2] states that $\overline{AB}$ is bisected at $\overline{CF}$. □

[1.10] Exercises

2. Extend $\overline{CD}$ to $\overleftrightarrow{CD}$. Prove that every point equally distant from the points A and B are points on $\overleftrightarrow{CD}$.

PROOF. Extend $\overline{CD}$ to $\overleftrightarrow{CD}$. We claim that every point equally distant from the points A and B are points on $\overleftrightarrow{CD}$.

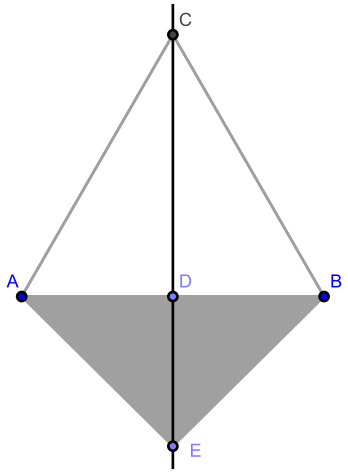


FIGURE 9.1.12. [1.10, #2]

By the proof of [1.10], $\triangle ACB$ is isosceles. By [Cor. 1.9.1], $\overleftrightarrow{CD}$ is an Axis of Symmetry to $\triangle ACB$. It follows that $\overleftrightarrow{CD}$ is an Axis of Symmetry to any triangle with vertices A, B, and E, where E is any point on $\overleftrightarrow{CD}$ except D itself. Hence, $\overline{AE} = \overline{EB}$ for any point E on $\overleftrightarrow{CD}$. Since $\overline{AD} = \overline{DB}$ by [1.10], we have proven that every point on $\overleftrightarrow{CD}$ is equally distant from A and B.

Suppose a point K not on $\overleftrightarrow{CD}$ exists where $\overline{AK} = \overline{BK}$. It follows that $\triangle ABE$ is distinct from $\triangle ABK$ where the triangles share side AB, $AE = AK$, and $BE = BK$. This contradicts [1.7]. Hence, no such K exists, and we have proven our claim. □

[1.11] Exercises

1. Prove that the diagonals of a rhombus bisect each other perpendicularly.

PROOF. Construct the figure from [1.11], and also construct the equilateral triangle $\triangle DEG$ where G lies on the opposite side of $\overline{AB}$ from F. By [1.8], $\triangle DEF \cong \triangle DEG$. Construct the segment $\overline{GC}$; by the proof of [1.11], $GC \perp AB$. We claim that:

(1) $\boxdot FEGD$ is a rhombus

(2) $\overline{GF}$ and $\overline{DE}$ are its diagonals

(3) $\overline{GF}$ and $\overline{DE}$ bisect each other

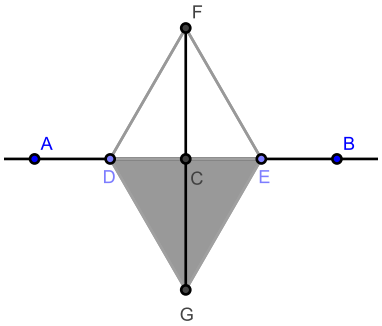


FIGURE 9.1.13. [1.11, #1]

Consider $\triangle DCF$ and $\triangle DCG$: since $\triangle DEF \cong \triangle DEG$, $DF = DG$ and $\angle CDG = \angle CDF$; the triangles also share side DC. By [1.4], $\triangle DCF \cong \triangle DCG$. Similarly, we can show that $\triangle ECF \cong \triangle ECG$.

Consider $\triangle DCF$ and $\triangle ECF$: since $\triangle DFE$ is equilateral, $DF = EF$; since C is the midpoint of $\overline{DE}$, $\overline{DC} = \overline{CE}$; finally, each triangle shares side CF. By [1.8], $\triangle DCF \cong \triangle ECF$. Similarly, we can show that $\triangle DCG \cong \triangle ECG$.

By the above, it follows that

$$\triangle DCF \cong \triangle DCG \cong \triangle ECF \cong \triangle ECG$$

Hence, $\overline{FD} = \overline{FE} = \overline{GD} = \overline{GE}$, and so $\boxdot FEGD$ is a rhombus (claim 1).

Clearly, $\overline{GC} \oplus \overline{CF} = \overline{GF}$ where $\overline{GF}$ is a diagonal of $\boxdot FEGD$; similarly, $\overline{DE}$ is the other diagonal of $\boxdot FEGD$ (claim 2).

Since $\overline{DC} = \overline{CE}$, we have that $\overline{GF}$ bisects $\overline{DE}$. Similarly, we can show that $\overline{DE}$ bisects $\overline{GF}$ (claim 3). This completes the proof. $\square$

[1.11] Exercises

3. Find a point on a given line that is equally distant from two given points.

PROOF. Let $\overleftrightarrow{AB}$ be a given line, and let C, D be points not on $\overleftrightarrow{AB}$. We wish to find a point F on $\overleftrightarrow{AB}$ such that $\overline{FC} = \overline{FD}$.

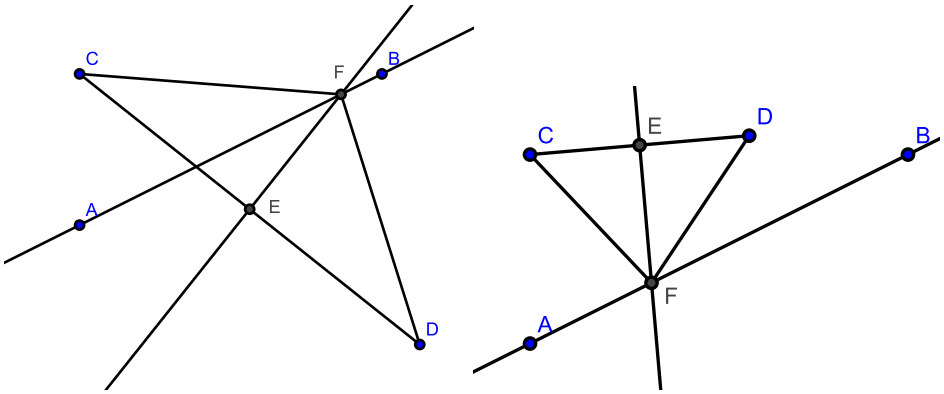


FIGURE 9.1.14. [1.11, #3]

Construct the segment $\overline{CD}$; by [1.10], locate its midpoint, E. Construct $\overleftrightarrow{FE}$ such that $\overline{CD} \perp \overleftrightarrow{FE}$ and F is a point on $\overleftrightarrow{AB}$. We claim that F is equally distant from C and D.

Consider $\triangle CEF$ and $\triangle DEF$: $CE = DE$ by construction, $\angle CEF = \angle DEF$ by construction, and the triangles share side EF. By [1.4], $\triangle CEF \cong \triangle DEF$. Hence, $\overline{CF} = \overline{DF}$. It follows that F is equally distant from C and D. $\square$

[1.11] Exercises

5. Find a point that is equidistant from three given points. (Hint: you are looking for the circumcenter of the triangle formed by the points.)

PROOF. Let A, B, and C be arbitrary points and construct $\triangle ABC$. By [1.10], we may locate the midpoints of each side of $\triangle ABC$ where F is the midpoint of AC, E is the midpoint of BC, and D is the midpoint of AB.

Construct $\overleftrightarrow{FG}$ such that $\overleftrightarrow{FG} \perp \overline{AC}$. Similarly, construct $\overleftrightarrow{EG}$ such that $\overleftrightarrow{EG} \perp \overline{BD}$ and $\overleftrightarrow{DG}$ such that $\overleftrightarrow{DG} \perp \overline{AB}$. Since none of the pairs of sides of $\triangle ABC$ are parallel, $\overleftrightarrow{FG}$, $\overleftrightarrow{DG}$, and $\overleftrightarrow{EG}$ have a common point of intersection: G. We claim that G is equidistant from A, B, and C.

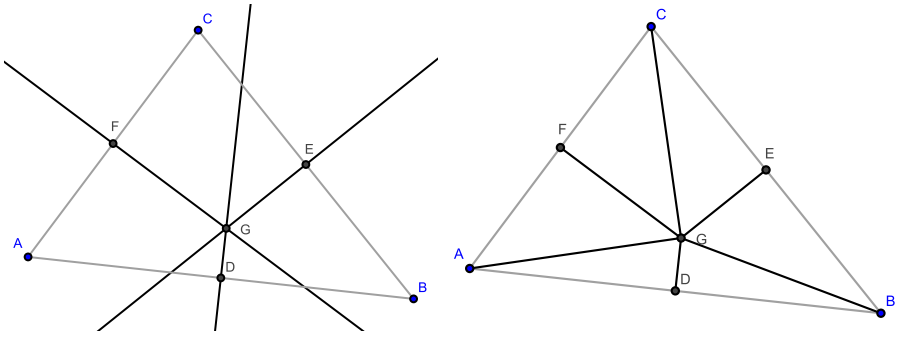


FIGURE 9.1.15. [1.10, #5]

Construct $\overline{AG}$, $\overline{BG}$, and $\overline{CG}$. Consider $\triangle ADG$ and $\triangle BDG$: $AD = BD$, $\angle ADG = \angle BDG$, and the triangles share side DG. By [1.4], $\triangle ADG \cong \triangle BDG$. Hence, $\overline{AG} = \overline{BG}$.

Similarly, consider $\triangle AFG$ and $\triangle CFG$: by an argument similar to the above, $\triangle AFG \cong \triangle CFG$, and so $\overline{AG} = \overline{CG}$. It follows that

$$\overline{AG} = \overline{BG} = \overline{CG}$$

and therefore G is equidistant from points A, B, C. □

[1.12] Exercises

1. Prove that circle $\bigcirc C$ cannot meet $\overleftrightarrow{AB}$ at more than two points.

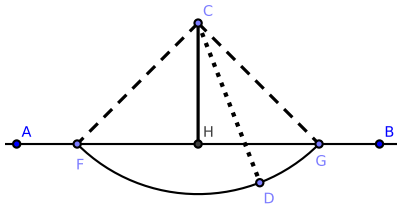


FIGURE 9.1.16. [1.12, #1]

PROOF. Suppose that $\bigcirc C$ intersects $\overleftrightarrow{AB}$ at more than two points. If the third point lies between points F and G, then the radius of $\bigcirc C$ must decrease in length; this contradicts the definition of a radius (that it must have a fixed length).

Similarly, if the third point lies to the left of F or to the right of G, the radius of $\bigcirc C$ must increase in length, resulting in a similar contradiction.

Hence, $\bigcirc C$ cannot intersect $\overleftrightarrow{AB}$ at more than two points. □

[1.19] Exercises

3. Prove that three equal and distinct segments cannot be constructed from the same point to the same line.

PROOF. Construct $\overleftrightarrow{AB}$ where A and B are arbitrary, distinct points; also construct C which is not on $\overleftrightarrow{AB}$. Construct $\overline{CA}$, $\overline{CB}$, and $\overline{CD}$ such that $\overline{CA} = \overline{CB}$. We claim that $\overline{CA} = \overline{CB} \neq \overline{CD}$.

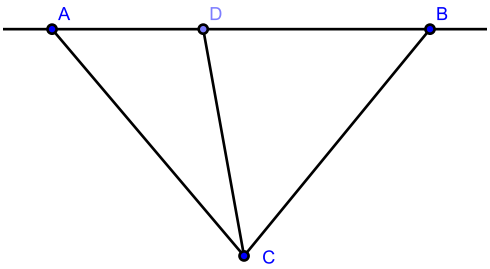


TABLE 1. [1.19, #3]

Suppose that $\overline{CA} = \overline{CB} = \overline{CD}$, and consider $\triangle ACD$: since $\overline{CA} = \overline{CD}$, $\angle CAD = \angle CDA$. Similarly, $\angle CDB = \angle CBD$. Since $\angle CDA$ and $\angle CDB$ are supplementary, they are right angles, and so $\overline{CD} \perp \overleftrightarrow{AB}$. It follows that $\angle CDA = \angle CDB$, and so

$$\angle CAD = \angle CDA = \angle CDB = \angle CBD$$

Since each are right angles, we must have that $\overline{CA} \perp \overleftrightarrow{AB}$ and $\overline{CB} \perp \overleftrightarrow{AB}$; this contradicts the construction of $\triangle ABC$. Therefore $\overline{CA} = \overline{CB} \neq \overline{CD}$, and our claim is proven. $\square$

[1.19] Exercises

5. If $\triangle ABC$ is a triangle such that side $AB \leq AC$, then a segment $\overline{AG}$, constructed from A to any point G on side BC, is less than AC.

PROOF. Construct $\triangle ABC$ where $AB < AC$. Construct $\overline{AG}$ where G is a point on side BC (other than B and C). We claim that $AG < AC$.

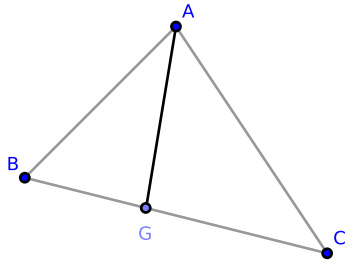


FIGURE 9.1.17. [1.19, #5]

Since $AB < AC$, by [1.16] $\angle ACB < \angle ABC$. By [1.19], $\angle ABC < \angle AGC$, and so $\angle ACB < \angle AGC$. Notice that $\angle ACB = \angle ACG$.

Consider $\triangle AGC$: since $\angle ACG < \angle AGC$, by [1.19] $AG < AC$.

If $\triangle ABC$ where $AB = AC$, the proof follows *mutatis mutandis*. □

[1.20] Exercises

5. The perimeter of a quadrilateral is greater than the sum of its diagonals.

PROOF. Suppose that $ABCD$ is a quadrilateral with diagonals $\overline{AC}$ and $\overline{BD}$. We claim that

$$\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA} > \overline{AC} + \overline{BD}$$

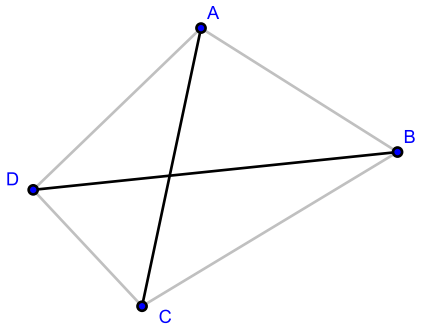


FIGURE 9.1.18. [1.20, #5]

By [1.20], we have that

$$\begin{aligned} AD + DC &> AC \\ AB + BC &> AC \\ AD + AB &> BD \\ BC + CD &> BD \end{aligned}$$

Or,

$$\begin{aligned} AB + BC + CD + DA &> 2 \cdot AC \\ AB + BC + CD + DA &> 2 \cdot BD \\ &\Longrightarrow \\ 2 \cdot (AB + BC + CD + DA) &> 2 \cdot (AC + BD) \\ &\Longrightarrow \\ AB + BC + CD + DA &> AC + BD \end{aligned}$$

which proves our claim. □

[1.20] Exercises

6. The sum of the lengths of the three medians of a triangle is less than $\frac{3}{2}$ times its perimeter.

PROOF. Construct $\triangle ABC$ with medians $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$. We claim that

$$\overline{AF} + \overline{BE} + \overline{CD} < \frac{3}{2} \cdot (\overline{AB} + \overline{BC} + \overline{AC})$$

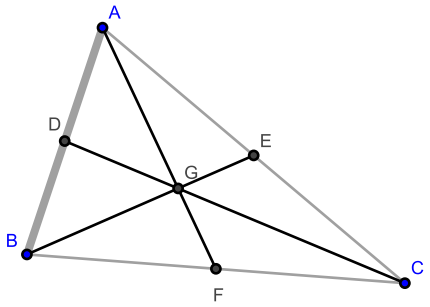


FIGURE 9.1.19. [1.20, #6]

Consider $\triangle ABE$: by [1.20], we have

$$\overline{AE} + \overline{AB} > \overline{BE}$$

Similarly, in $\triangle DBC$, we have

$$\overline{BD} + \overline{BC} > \overline{CD}$$

and in $\triangle ACF$, we have

$$\overline{AC} + \overline{CF} > \overline{AF}$$

Recall that $\overline{AE} = \frac{1}{2}\overline{AC}$, $\overline{BD} = \frac{1}{2}\overline{AB}$, and $\overline{CF} = \frac{1}{2}\overline{BC}$. Adding each inequality, we have that

$$\begin{aligned} \overline{AB} + \overline{BC} + \overline{AC} + \overline{AE} + \overline{BD} + \overline{CF} &> \overline{AF} + \overline{BE} + \overline{CD} \\ \overline{AB} + \overline{BC} + \overline{AC} + \tfrac{1}{2}\overline{AB} + \tfrac{1}{2}\overline{BC} + \tfrac{1}{2}\overline{AC} &> \overline{AF} + \overline{BE} + \overline{CD} \\ \tfrac{3}{2}(\overline{AB} + \overline{BC} + \overline{AC}) &> \overline{AF} + \overline{BE} + \overline{CD} \end{aligned}$$

and our claim is proven. □

[1.23] Exercises

1. Construct a triangle given two sides and the angle between them.

PROOF. Suppose we have arbitrary segments $\overline{AB}$ and $\overline{CD}$ and an arbitrary angle $\angle EFG$. We claim that we can construct $\triangle HMN$ from $\overline{AB}$, $\overline{CD}$, and $\angle EFG$.

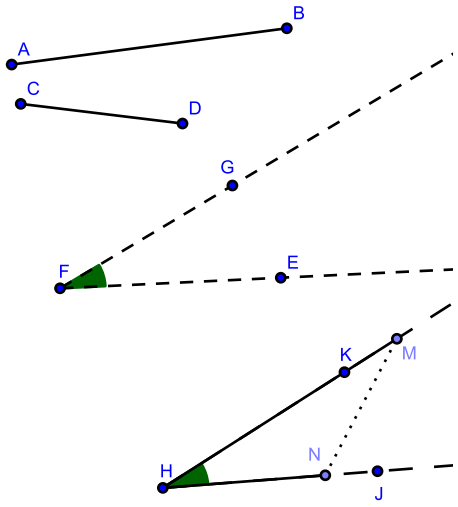


FIGURE 9.1.20. [1.23, #1]

By [1.23], construct rays $\overrightarrow{HJ}$ and $\overrightarrow{HK}$ such that $\angle JHK = \angle EFG$. Construct $\overline{HM}$ on $\overrightarrow{HJ}$ and $\overline{HN}$ on $\overrightarrow{HK}$ such that $\overline{AB} = \overline{HM}$ and $\overline{CD} = \overline{HN}$. Construct $\overline{MN}$. Notice that $\triangle MNH$ has sides equal in length to segments $\overline{AB}$ and $\overline{CD}$ and contains an angle equal in measure to $\angle EFG$. Therefore, our claim is proved. □

[1.26] Exercises

2. If the line which bisects the vertical angle of a triangle also bisects the base, the triangle is isosceles.

PROOF. Construct $\triangle ABC$ with $\overline{AD}$ such that $\angle BAC = \angle BAD + \angle CAD$, $\angle BAD = \angle CAD$, and $\overline{BD} = \overline{DC}$. We wish to prove that $\angle CBA = \angle BCA$.

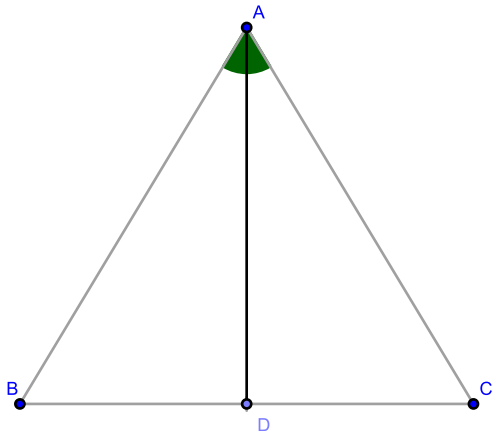


FIGURE 9.1.21. [1.26, #2]

Consider $\triangle ABD$ and $\triangle ACD$. By [1.38], the triangles have equal area, and so $\overline{AD}$ bisects $\triangle ABC$. By [Def. 1.35], $\overline{AD}$ is the Axis of Symmetry of $\triangle ABC$; hence, $\angle CBA = \angle BCA$, and we have proven our claim. □

[1.26] Exercises

6. Prove that if two right triangles have equal hypotenuses and that if a side of one is equal in length to a side of the other, then the triangles they are congruent. (Note: this proves the special case of Side-Side-Angle congruency for right triangles.)

PROOF. Suppose that $\triangle ABC$ and $\triangle DEF$ are right triangles where $\angle ACB$ and $\angle DFE$ are right angles, the hypotenuses AB and DE are equal in length, and $AC = DF$. (The proof follows *mutatis mutandis* if $BC = EF$.) We wish to prove that $\triangle ABC \cong \triangle DEF$.

By superposition, move $\triangle DEF$ so that side AC coincides with side DF such that $A = D$ and $C = F$.

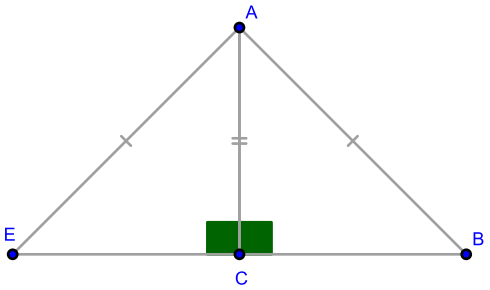


FIGURE 9.1.22. [1.26, #6] after moving the triangles

Consider $\triangle AEB$. Since $AE = AB$, $\angle AEB = \angle ABE$ by [1.5].

Now consider $\triangle AEC$ and $\triangle ACB$. Since $\angle ACE = \angle ACB$, $\angle AEC = \angle ABC$, and $AE = AB$, by [1.26], $\triangle AEC \cong \triangle ABC$. This proves that $\triangle DEF = \triangle AEC$. $\square$

[1.29] Exercises

2. Construct $\overleftrightarrow{AB}$ containing the point C and $\overleftrightarrow{EF}$ containing the point D such that $\overleftrightarrow{AB} \parallel \overleftrightarrow{EF}$. Construct $\overline{CH}$ and $\overline{CJ}$ such that $\overline{CJ}$ bisects $\angle ACD$ and $\overline{CH}$ bisects $\angle BCD$. Prove that $\overline{DH} = \overline{DJ}$.

PROOF. Our hypothesis is stated above; we claim that $\overline{DH} = \overline{DJ}$.

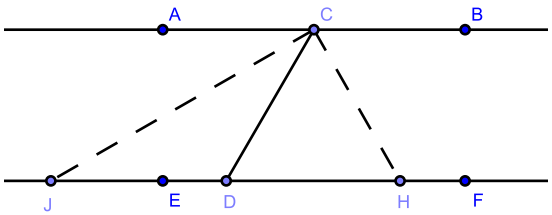


FIGURE 9.1.23. [1.29, #2]

Construct $\overleftrightarrow{JK}$ and $\overleftrightarrow{HL}$ such that $\overleftrightarrow{JK} \parallel \overline{CD}$ and $\overline{CD} \parallel \overleftrightarrow{HL}$.

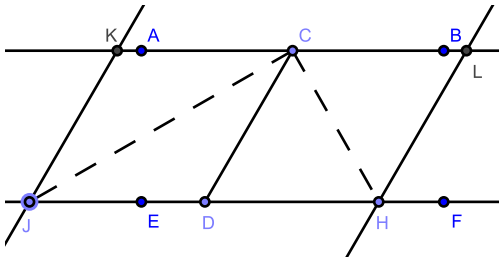


FIGURE 9.1.24. [1.29, #2] after additional construction

Since $\overline{CH}$ is a bisector of $\angle BCD$, $\angle BCH = \angle DCH$. Since $\overline{CD} \parallel \overleftrightarrow{HL}$, $\angle DCH = \angle CHL$. And since $\overleftrightarrow{AB} \parallel \overleftrightarrow{EF}$, $\angle BCH = \angle DHC$. This gives us

$$\angle BCH = \angle DCH = \angle DHC = \angle CHL$$

Consider $\triangle CDH$ and $\triangle CLH$: all angles are equal, $CD = BH$, and $DH = CB$. By [1.4], $\triangle CDH \cong \triangle CLH$, and so

$$CD = CL = HL = DH$$

By similar reasoning,

$$CD = KJ = CK = DJ$$

Therefore, $DJ = DH$, which proves our claim. □

[1.29] Exercises

5. Two lines passing through a point which is equidistant from two parallel lines intercept equal segments on the parallels.

PROOF. Construct $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ such that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Construct $\overleftrightarrow{LM}$ such that $\overleftrightarrow{AB} \perp \overleftrightarrow{LM}$ and bisect $\overleftrightarrow{LM}$ at G (so $\overline{GL} = \overline{GM}$) [1.10]. Construct arbitrary lines $\overleftrightarrow{HJ}$ and $\overleftrightarrow{IK}$ such that each passes through G. We claim that $\overline{HI} = \overline{JK}$.

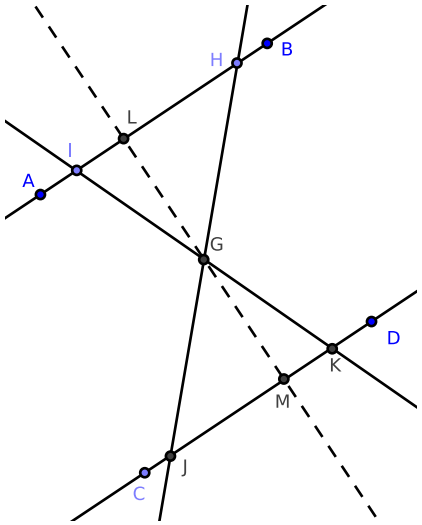


FIGURE 9.1.25. [1.29, #5]

Consider $\triangle GLI$ and $\triangle GMK$: $\angle LGI = \angle MGK$ by [1.15]; $GL = GM$ by construction; $\angle GMK = \angle GLI$ by construction. By [1.26], $\triangle GLI \cong \triangle GMK$, and so $GI = GK$.

Now consider $\triangle GHI$ and $\triangle GJK$: $\angle HGI = \angle JGK$ by [1.15]; since $AB \parallel CD$, $\angle GIH = \angle GKJ$ by [1.29, Cor. 1]; by the above, $GI = GK$. Again by [1.26], we have that $\triangle GHI \cong \triangle GJK$. Therefore, $HI = JK$, which proves our claim. $\square$

[1.31] Exercises

1. Given the altitude of a triangle and the base angles, construct the triangle.

PROOF. Suppose we are given altitude $h = \overline{AB}$ and base angles α and β. We wish to construct the triangle with altitude h and base angles α and β.

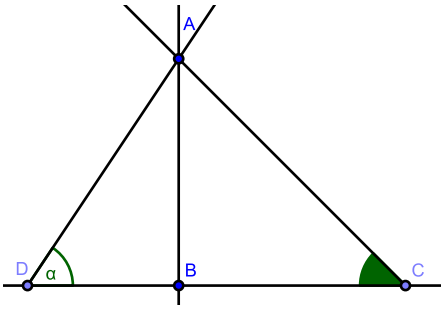


FIGURE 9.1.26. [1.31, #1]

Extend h to $\overleftrightarrow{AB}$, and construct $\overleftrightarrow{BC}$ such that $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$ [Cor. 1.11.1]. Let α be the angle on the left half of the base. Construct $\overrightarrow{DA}$ such that D is a point on $\overleftrightarrow{BC}$ and $\angle ADB = \alpha$. Similarly, construct $\overrightarrow{CA}$ such that $\angle DCA = \beta$.

Clearly, $\triangle ACD$ contains angles α and β and has altitude h; this is the triangle we wished to construct. □

[1.31] Exercises

5. Through two given points on two parallel lines, construct two segments forming a rhombus with given parallels.

PROOF. Construct parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ where B and C are the given points. We wish to construct the rhombus $\boxdot FBGC$ by constructing the segments $\overline{FC}$ and $\overline{BG}$.

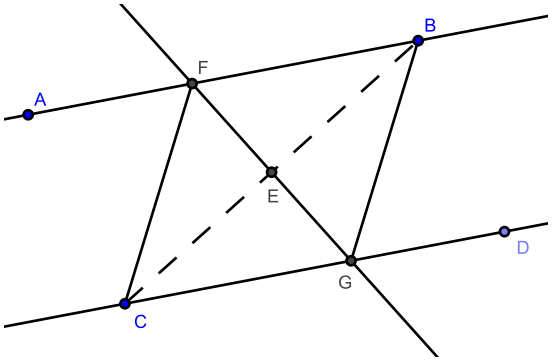


FIGURE 9.1.27. [1.31, #5]

Locate E, the midpoint of $\overleftrightarrow{BC}$ [1.10]. Construct $\overleftrightarrow{FG}$ such that $\overleftrightarrow{FG} \perp \overline{BC}$. Construct segments $\overline{FC}$ and $\overline{BG}$. We claim that $\boxdot FBGC$ is the required rhombus.

By construction, $\boxdot FBGC$ is a parallelogram.

Consider $\triangle CEG$ and $\triangle BEF$: by construction, $\overline{BE} = \overline{CE}$; by [1.15], $\angle CEG = \angle BEF$; by [Cor. 1.29.1], $\angle BFE = \angle CGE$. Then [1.26] states that $\triangle CEG \cong \triangle BFE$, and so $\overline{BF} = \overline{CG}$ and $\overline{EF} = \overline{EG}$.

Now consider $\triangle FEC$ and $\triangle GEB$: we have that $\overline{BE} = \overline{CE}$, $\overline{EF} = \overline{EG}$, and $\angle FEC = \angle BEG$ by [1.15]. By [1.4], $\triangle FEC \cong \triangle BEG$. Hence, $\overline{FC} = \overline{BG}$.

Since

$$\angle CEG = \angle GEB = \angle BEF = \angle FEC$$

(because they are all right angles), by [1.4] again, we have that

$$\triangle CEG \cong \triangle BFE \cong \triangle FEC \cong \triangle BEG$$

and so

$$\overline{BF} = \overline{BG} = \overline{FC} = \overline{CG}$$

Thus by [Def 1.29], $\boxdot FBGC$ is a rhombus. □

[1.32] Exercises.

3. If the line which bisects an external vertical angle of a triangle is parallel to the base of the triangle, then the triangle is isosceles.

PROOF. Construct $\triangle ABC$ with external vertical angle $\angle ACE$ such that $\overleftrightarrow{CD}$ bisects $\angle ACE$ and $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$. We claim that $\triangle ABC$ is isosceles.

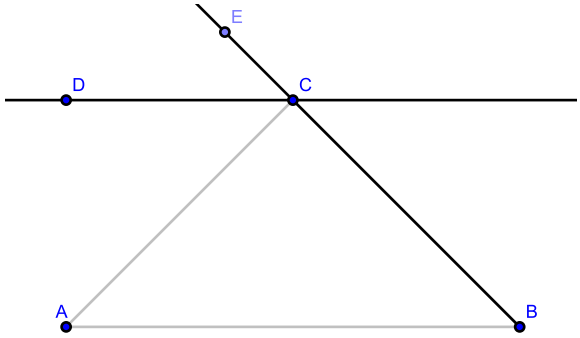


FIGURE 9.1.28. [1.32, #3]

By [1.29, Cor. 1], $\angle DCE = \angle ABC$. Also by By [1.29, Cor. 1], $\angle DCA = \angle BAC$. But $\angle DCE = \angle DCA$ by hypothesis, and so $\angle ABC = \angle BAC$. By [1.6], $\triangle ABC$ is isosceles. □

[1.32] Exercises

5. The three perpendicular bisectors of a triangle are concurrent.

PROOF. Construct the following: $\triangle ABC$, altitudes $\overline{AG}$ and $\overline{CF}$, and $\overline{DB}$ where D is the intersection of $\overline{AG}$ and $\overline{CF}$. Extend $\overline{DB}$ to $\overline{BE}$ where E is a point on $\overline{AC}$. We claim that $\overline{BE}$ is an altitude of $\triangle ABC$.

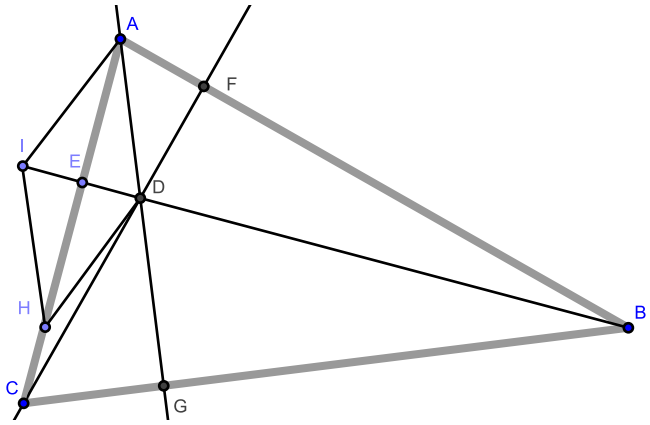


FIGURE 9.1.29. [1.32, #5]

Construct segments $\overline{HD}$, $\overline{HI}$, and $\overline{AI}$ such that

$$\overline{AD} = \overline{HD} = \overline{HI} = \overline{AI}$$

By [Def. 1.29], $\square ADHI$ is a rhombus. By [1.11, #1], the diagonals of a rhombus bisect each other perpendicularly. Since AH is a diagonal of $\square ADHI$, AH is bisected perpendicularly by DI at point E, and so $\angle AED$ equals one right angle. If $\angle EDB$ equals two right angles, then we will have proven our claim.

By the construction of $\angle BDA$, we have that $\angle EDB = \angle BDA + \angle EDA$ and by [1.32] $\angle BDA = \angle DEA + \angle EAD$. Hence

$$\begin{aligned} \angle EDB &= \angle BDA + \angle EDA \\ &= \angle DEA + \angle EAD + \angle EDA \\ &= \text{two right angles} \end{aligned}$$

since $\angle DEA$, $\angle EAD$, and $\angle EDA$ are the interior angles of $\triangle EDA$.

This proves our claim. $\square$

[1.32] Exercises

6. The bisectors of two adjacent angles of a parallelogram stand at right angles.

PROOF. Construct a parallelogram; if the parallelogram is a rhombus, the result follows from [1.11, #1].

Otherwise, construct $\boxdot ABCD$ such that $\boxdot ABCD$ is not a rhombus. Construct $\angle DAF$ such that $\angle DAF = \frac{1}{2}\angle DAB$ and $\angle ABE$ such that $\angle ABE = \frac{1}{2}\angle ABC$.

Extend the following segments to lines: $\overline{AD}$, $\overline{AF}$, $\overline{BE}$, $\overline{BC}$. Let H be the intersection of $\overleftrightarrow{BC}$ and $\overleftrightarrow{AF}$. Construct $\overline{HG}$ such that $\overline{HG} \parallel \overline{AB}$ and G is on $\overleftrightarrow{AD}$. Since $\overline{BC} \parallel \overline{AD}$, $\overline{BH} \parallel \overline{AG}$. Hence, $\boxdot ABHG$ is a parallelogram. We claim that $\boxdot ABHG$ is a rhombus.

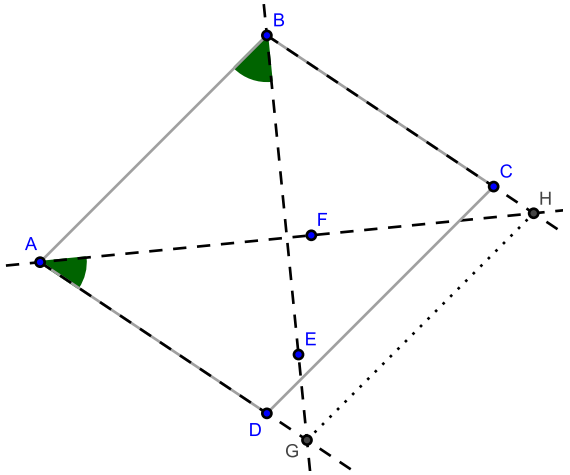


FIGURE 9.1.30. [1.32, #6]

By construction, $\angle GAH = \angle BAH$. By [1.29, Cor. 1], $\angle GAH = \angle AHB$ and $\angle BAH = \angle AHG$. Hence

$$\angle GAH = \angle BAH = \angle AHB = \angle AHG$$

Consider $\triangle ABH$ and $\triangle AGH$. Since $\angle BAH = \angle AHB$, $\triangle ABH$ is isosceles. A similar argument holds for $\triangle AGH$. Given the equality of the four base angles above and that the two triangles share base AH, by [1.26] $\triangle ABH \cong \triangle AGH$. Hence, we have that

$$\overline{AB} = \overline{BH} = \overline{HG} = \overline{GA}$$

By [Def 1.29], $\boxdot ABHG$ is a rhombus. □

[1.33] Exercises

1. Prove that if two segments $\overline{AB}$, $\overline{BC}$ are respectively equal and parallel to two other segments $\overline{DE}$, $\overline{EF}$, then the segment $\overline{AC}$ joining the endpoints of the former pair is equal in length to the segment $\overline{DF}$ joining the endpoints of the latter pair.

PROOF. Construct segments $\overline{AB}$, $\overline{BC}$, $\overline{DE}$, $\overline{EF}$ such that $\overline{AB} = \overline{DE}$, $\overline{BC} = \overline{EF}$, $\overline{AB} \parallel \overline{DE}$, and $\overline{BC} \parallel \overline{EF}$. Construct segments $\overline{AC}$ and $\overline{DF}$. We wish to show $\overline{AC} = \overline{DF}$.

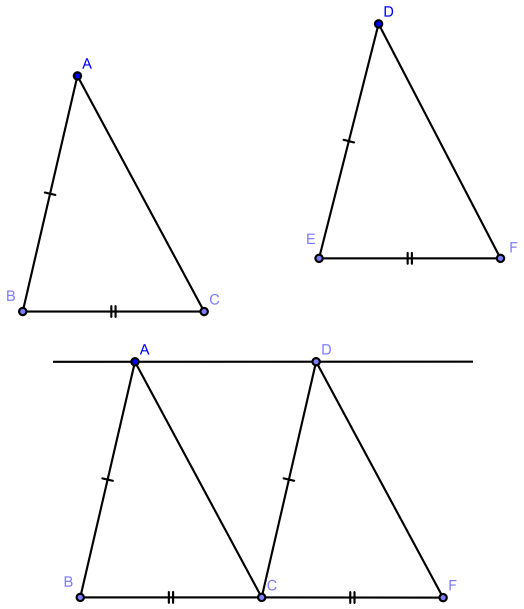


FIGURE 9.1.31. [1.33, #1]

Consider $\triangle ABC$ and $\triangle DEF$, and move $\triangle ABC$ such that $C = E$. Construct $\overleftrightarrow{AD}$ such that $\overleftrightarrow{AD} \parallel \overleftrightarrow{BF}$.

Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{BF}$, $\angle ABC = \angle DCF$ by [Cor. 1.29.1]. Since $\overline{AB} = \overline{DC}$ and $\overline{BC} = \overline{CF}$, by [1.4] we have that $\triangle ABC \cong \triangle DCF$. Thus, $\overline{AC} = \overline{DF}$, which proves our claim. $\square$

[1.34] Exercises

1. Show that the diagonals of a parallelogram bisect each other.

PROOF. Consider $\boxdot ABCD$ and diagonals $\overline{AD}$, $\overline{BC}$. Let point E be the intersection of $\overline{AD}$ and $\overline{BC}$. We claim that $\overline{AE} = \overline{ED}$ and $\overline{CE} = \overline{EB}$.

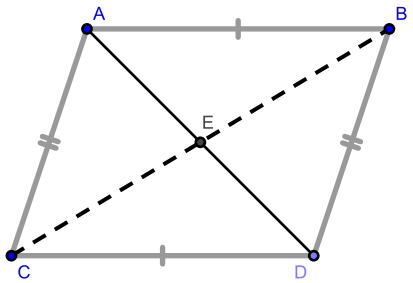


FIGURE 9.1.32. [1.34], #1

Since $\overline{AB} \parallel \overline{CD}$, $\angle BCD = \angle CBA$. Similarly, we have that $\angle CDA = \angle DAB$. Consider $\triangle ECD$ and $\triangle AEB$: since $\angle ECD = \angle EBA$, $\angle EDC = \angle EAB$, and $\overline{CD} = \overline{AB}$, by [1.26] we have that $\triangle ECD \cong \triangle AEB$. Hence, $\overline{AE} = \overline{ED}$ (claim 1).

A similar argument shows that $\overline{CE} = \overline{EB}$, *mutatis mutandis* (claim 2). This proves our claim. □

[1.34] Exercises

2. If the diagonals of a parallelogram are equal, each of its angles are right angles.

PROOF. Construct $\boxdot ABCD$ as in [1.34, #1] and suppose that $\overline{AD} = \overline{BC}$. We wish to show that

$$\angle CAB = \angle ABD = \angle BDC = \angle ACD = \frac{\pi}{2} \text{ radians}$$

in measure.

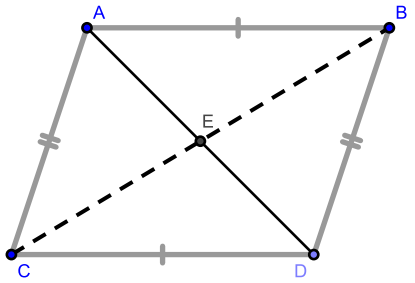


FIGURE 9.1.33. [1.34], #2

Consider $\triangle ECD$, $\triangle EDB$, $\triangle EBA$, and $\triangle EAC$; by [1.34, #1] and our hypothesis, these are isosceles triangles where

$$\overline{EA} = \overline{EB} = \overline{ED} = \overline{EC}$$

By [1.32, #6], $\angle CED = \angle DEB = \angle BEA = \angle AEC = \frac{\pi}{2}$ radians in measure. By [1.4],

$$\triangle ECD \cong \triangle EDB \cong \triangle EBA \cong \triangle EAC$$

Consider $\triangle ECD$: since $\angle CED = \frac{\pi}{2}$ radians, by [1.32] $\angle ECD + \angle EDC = \frac{\pi}{2}$ radians. Since $\triangle ECD$ is isosceles, $\angle ECD = \angle ECD$; therefore, each equals $\frac{\pi}{4}$ radians.

Consider $\angle ACE$ and $\angle ECD$: by the above, they are equal, and their sum is $\frac{\pi}{2}$. Since $\angle ACD = \angle ACE + \angle DCE$, and

$$\angle CAB = \angle ABD = \angle BDC = \angle ACD$$

our claim is proven. □

[1.37] Exercises

1. If two triangles of equal area stand on the same base but on opposite sides of the base, the segment connecting their vertices is bisected by the base or its extension.

PROOF. Suppose we have $\triangle ABG$ and $\triangle ABI$ such that both triangles share base $\overline{AB}$ and that G stands on the opposite side of $\overline{AB}$ than I. We claim that the segment $\overline{GI}$ is bisected by $\overleftrightarrow{AB}$.

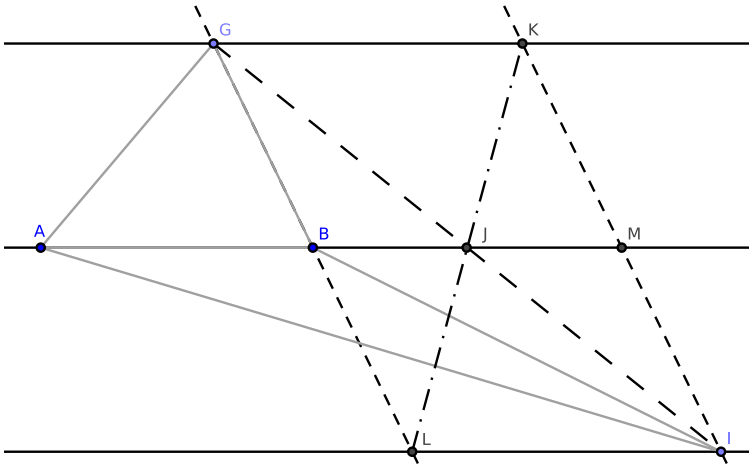


FIGURE 9.1.34. [1.37, #1]

Extend $\overline{AB}$ to $\overleftrightarrow{AB}$, and let J be the point where $\overline{GI}$ intersects $\overleftrightarrow{AB}$. Construct lines $\overleftrightarrow{GK}$ and $\overleftrightarrow{LI}$ such that $\overleftrightarrow{GK} \parallel \overleftrightarrow{AB}$ and $\overleftrightarrow{LI} \parallel \overleftrightarrow{AB}$. By [1.30], $\overleftrightarrow{GK} \parallel \overleftrightarrow{LI}$. Also construct lines $\overleftrightarrow{GB}$ and $\overleftrightarrow{KI}$ such that $\overleftrightarrow{GB} \parallel \overleftrightarrow{KI}$. Hence, $\boxdot GKIL$ is a parallelogram.

Construct $\overline{JL}$ and $\overline{JK}$. If $\angle LJK = \pi$ radians in measure, then $\overline{LK}$ is a segment. Notice that $\angle LJK = \angle LJG + \angle GJK$ and that $\angle GJK = \angle LJI$ by [1.15]. Since $\angle GJI = \angle LJG + \angle LJI$ and $\overline{GI}$ is a segment, it follows that $\angle GJI = \angle LJK = \pi$ radians in measure; hence, $\overline{LK}$ is a segment with J as a point. Furthermore, $\overline{LK}$ and $\overline{GJ}$ are diagonals of $\boxdot GKIL$.

By [1.34, #1], $\overline{GJ} = \overline{JI}$. Since $\overline{GI} = \overline{GJ} \oplus \overline{JI}$, the proof follows. $\square$

[1.38] Exercises

1. Every median of a triangle bisects the triangle.

PROOF. Construct $\triangle ABC$ where $\overline{AF}$ is the median of side BC. We claim that $\triangle ABF = \triangle ACF$.

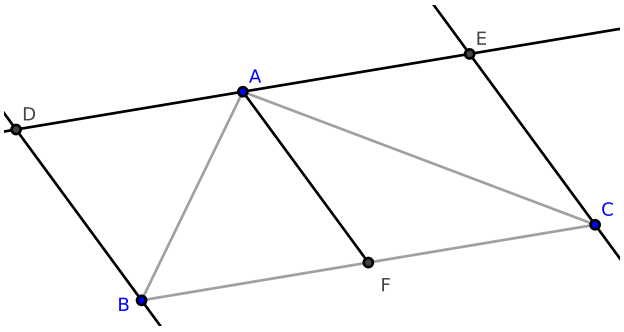


FIGURE 9.1.35. [1.38, #1]

Construct $\overleftrightarrow{AE}$ such that $\overleftrightarrow{AE} \parallel \overline{BC}$. Clearly, $\triangle ABF$ and $\triangle ACF$ stand between the same parallels. Since $\overline{BF} = \overline{FC}$ by hypothesis, by [1.38] we have that $\triangle ABF = \triangle ACF$. This proves our claim. □

Note: we do **not** claim that the triangles are congruent, merely equal in area.

[1.38] Exercises

5. One diagonal of a quadrilateral bisects the other if and only if the diagonal also bisects the quadrilateral.

PROOF. Construct quadrilateral $ABCD$ and diagonals $\overline{AC}$ and $\overline{BD}$.

Claim 1: if diagonal $\overline{BD}$ bisects diagonal $\overline{AC}$, then $\overline{BD}$ bisects $ABCD$.

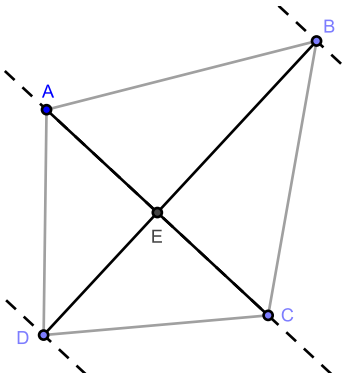


FIGURE 9.1.36. [1.38, #5]

Extend $\overline{AC}$ to $\overleftrightarrow{AC}$ and construct $\overleftrightarrow{DY}$ and $\overleftrightarrow{BZ}$ such that $\overleftrightarrow{DY} \parallel \overleftrightarrow{AC}$ and $\overleftrightarrow{BZ} \parallel \overleftrightarrow{AC}$; by [1.30], $\overleftrightarrow{DY} \parallel \overleftrightarrow{BZ}$.

Since $\overline{AE} = \overline{EC}$ by hypothesis, $\triangle ABE = \triangle CBE$ and $\triangle ADE = \triangle CDE$ by [1.38]. Since $\triangle ADB = \triangle ABE \oplus \triangle ADE$ and $\triangle CDB = \triangle CBE \oplus \triangle CDE$, it follows that $\triangle ADB = \triangle CDB$. Since $ABCD = \triangle ADB \oplus \triangle CDB$, we have that $ABCD$ is bisected by $\overline{BD}$.

Claim 2: if $ABCD$ is bisected by $\overline{BD}$, then $\overline{BD}$ bisects $\overline{AC}$.

Consider $\triangle ADB$ and $\triangle CDB$: the triangles stand on the same base (BD) but on opposite sides, and $\overline{AC}$ connects their vertices. By [1.37, #1], $\overline{BD}$ bisects $\overline{AC}$.

This proves our claim. □

[1.40] Exercises

1. Triangles with equal bases and altitudes are equal in area.

PROOF. Suppose we have two triangles with equal bases and with equal altitudes. Since the altitude of a triangle is the distance between the parallels which contain it, equal altitudes imply that the triangles stand between parallels which are equal distances apart. Therefore [1.37], [1.38], [1.39], and [1.40] prove the claim. □

[1.40] Exercises

2. The segment joining the midpoints of two sides of a triangle is parallel to the base, and the medians from the endpoints of the base to these midpoints will each bisect the original triangle. Hence, the two triangles whose base is the third side and whose vertices are the points of bisection are equal in area.

PROOF. Construct $\triangle ABC$ with midpoint D on side AB and midpoint E on side AC. Construct $\overline{DE}$, $\overline{DC}$, and $\overline{EB}$. We claim that:

1. $\overline{DE} \parallel \overline{BC}$
2. $\triangle ABE = \triangle CBE$
3. $\triangle ADC = \triangle BDC$
4. $\triangle BDC = \triangle CBE$

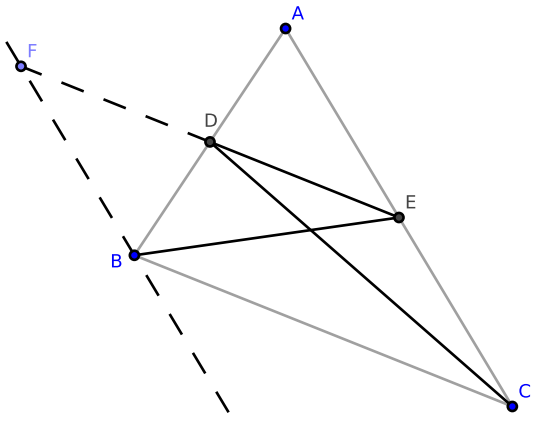


FIGURE 9.1.37. [1.40, #2]

Construct $\overline{DF}$ such that $\overline{DE} = \overline{DF}$ and $\overline{EF} = \overline{DE} \oplus \overline{DF}$; also construct $\overline{FB}$. Consider $\triangle ADE$ and $\triangle BDF$: by [1.15], $\angle ADE = \angle BDF$; by hypothesis, $\overline{AD} = \overline{BD}$; $\overline{DE} = \overline{DF}$ by construction. By [1.4], $\triangle ADE \cong \triangle BDF$, and so $\angle FBD = \angle DAE$. By [1.29, Cor. 1], $\overline{FB} \parallel \overline{AC}$.

Since $\triangle ADE \cong \triangle BDF$, $\overline{FB} = \overline{AE}$. Because $\overline{AE} = \overline{EC}$ by construction, $\overline{FB} = \overline{EC}$. By the above, $\overline{FB} \parallel \overline{CE}$; by [1.33] $\overline{EF} = \overline{BC}$ and $\overline{EF} \parallel \overline{BC}$. Hence, $\boxdot FECB$ is a parallelogram, and $\overline{DE} \parallel \overline{BC}$ (claim 1).

By [1.38, #1], $\overline{BE}$ bisects $\triangle ABC$, and so $\triangle ABE = \triangle CBE$ (claim 2). Similarly, $\overline{CD}$ bisects $\triangle ABC$, and so $\triangle ADC = \triangle BDC$ (claim 3).

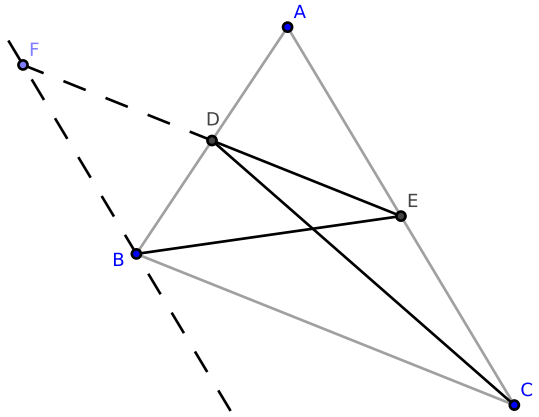


FIGURE 9.1.38. [1.40, #2]

By the above,

$$\triangle BDC = \frac{1}{2} \cdot \triangle ABC = \triangle CBE$$

and therefore $\triangle BDC = \triangle CBE$ (claim 4). This proves our claim.

□

COROLLARY. *Medians of triangles are half the length of the side they stand opposite to.*

[1.40] Exercises

4. The segments which connect the midpoints of the sides of a triangle divide the triangle into four congruent triangles.

PROOF. Suppose $\triangle ABC$ has midpoints D on side BC, E on side AC, and F on side AB. Construct segments $\overline{DE}$, $\overline{EF}$, and $\overline{DF}$. We claim that

$$\triangle AEF \cong \triangle ECD \cong \triangle FDB \cong \triangle DFE$$

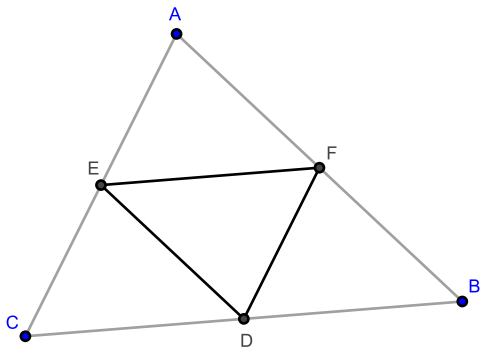


FIGURE 9.1.39. [1.40, #4]

By [1.40, #2], we have that $\overline{DE} \parallel \overline{AB}$, $\overline{DF} \parallel \overline{AC}$, and $\overline{EF} \parallel \overline{BC}$. By [1.29, Cor. 1], we have that:

$$\angle EDC = \angle DEF = \angle EFA = \angle DBF$$

$$\angle CED = \angle EAF = \angle DFB = \angle EDF$$

$$\angle ECD = \angle AEF = \angle FDB = \angle EFD$$

Since $\boxdot EFCD$ is a parallelogram, $\overline{EF} = \overline{CD}$. Since $\boxdot EFBD$ is a parallelogram, $\overline{EF} = \overline{BD}$, and so

$$\overline{EF} = \overline{CD} = \overline{DB}$$

By multiple applications of [1.26],

$$\triangle AEF \cong \triangle DEF \cong \triangle ECD \cong DBF$$

which proves our claim. □

[1.46] Exercises

1. Two squares have equal side-lengths if and only if the squares are equal in area.

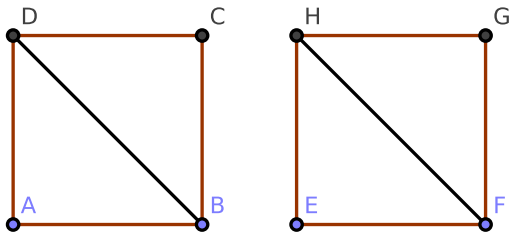


FIGURE 9.1.40. [1.46, #1]

PROOF. We claim the above.

Suppose $\boxdot ABCD$ and $\boxdot EFGH$ are squares where $AB = EF$. Construct $\overline{BD}$ and $\overline{FH}$. Notice that we have four triangles where each base is equal to the other three bases and each altitude is equal to the other three altitudes. By [1.40, #1], the triangles are equal in area. Since $\boxdot ABCD = \triangle ABD \oplus \triangle EFH$ and $\boxdot EFGH = \triangle CBD \oplus \triangle GFH$, $\boxdot ABCD = \boxdot EFGH$.

Now suppose that $\boxdot ABCD = \boxdot EFGH$ but $AB > EF$. It follows that $\triangle ABD$ has a longer altitude than $\triangle EFH$; by the contrapositive statement of [1.40, #1], $\triangle ABD \neq \triangle EFH$. Similarly, $\triangle CBD \neq \triangle GFH$. Since $\boxdot ABCD = \triangle ABD \oplus \triangle EFH$ and $\boxdot EFGH = \triangle CBD \oplus \triangle GFH$, $\boxdot ABCD \neq \boxdot EFGH$, a contradiction.

The proves our claim. □

[1.47] Exercises

4. Find a segment whose square is equal to the sum of the areas of two given squares.

PROOF. Let $\boxdot ABCD$ and $\boxdot EFGH$ be the given squares. We wish to construct $\overline{BF}$ such that $(\overline{BF})^2 = \boxdot ABCD + \boxdot EFGH$.

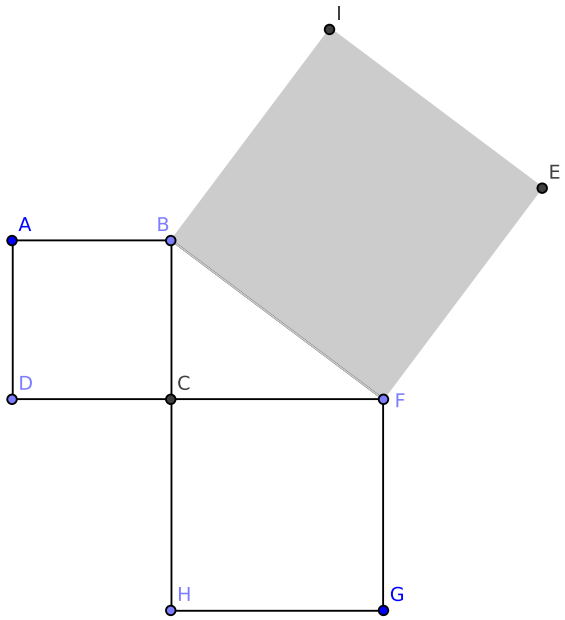


FIGURE 9.1.41. [1.47, #4]

Position $\boxdot ABCD$ and $\boxdot EFGH$ such that $C = E$ and $\overline{BC} \perp \overline{CF}$. Construct $\boxdot BFEI$ such that $\boxdot BFEI$ is a square with side-length $\overline{BF}$. By [1.47], $(\overline{BC})^2 + (\overline{CF})^2 = (\overline{BF})^2$. Since $(\overline{BC})^2 = \boxdot ABCD$ and $(\overline{CF})^2 = \boxdot EFGH$, $\overline{BF}$ is the required segment. □

[1.47] Exercises

10. Each of the triangles $\triangle AGK$ and $\triangle BEF$ formed by joining adjacent corners of the squares in [1.47] is equal in area to $\triangle ABC$.

PROOF. Construct the polygons as in [1.47] as well as $\overline{AG}$ and $\overline{EF}$. We claim that

$$\triangle KAG = \triangle ABC = \triangle BEF$$

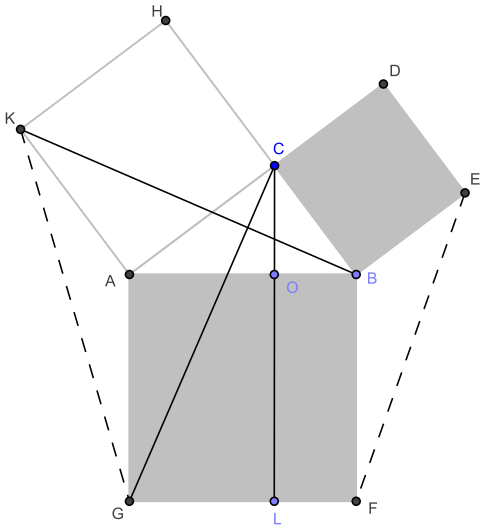


FIGURE 9.1.42. [1.47, #10]

Note that one right angle $= \frac{\pi}{2}$ radians. Consider $\triangle ABC$: notice that if $\angle ACB = \frac{\pi}{2}$ and $\angle BAC = \gamma$, then $\angle ABC = \frac{\pi}{2} - \gamma$. Since $\angle KAC = \angle GAB = \frac{\pi}{2}$ and

$$\angle KAG + \angle KAC + \angle BAC + \angle GAB \quad = \quad 2\pi$$

it follows that

$$\begin{aligned} \angle KAG &= 2\pi - \angle KAC - \angle BAC - \angle GAB \\ &= 2\pi - \frac{\pi}{2} - \gamma - \frac{\pi}{2} \\ &= \pi - \gamma \end{aligned}$$

Similarly

$$\begin{aligned} \angle FBE &= 2\pi - 2 \cdot \frac{\pi}{2} - \frac{\pi}{2} + \gamma \\ &= \frac{\pi}{2} + \gamma \end{aligned}$$

Recall that the general form of the equation of the area of a triangle is:

$$\text{Area} = \frac{1}{2}xy \cdot \sin\theta$$

where θ is the interior angle to sides x and y. So

$$\begin{aligned} \text{Area } \triangle KAG &= \tfrac{1}{2}AK \cdot AG \cdot \sin(\angle KAG) \\ &= \tfrac{1}{2}AC \cdot AB \cdot \sin(\pi - \gamma) \\ &= \tfrac{1}{2}AC \cdot AB \cdot \sin(\gamma) \end{aligned}$$

by the properties of the sine function.

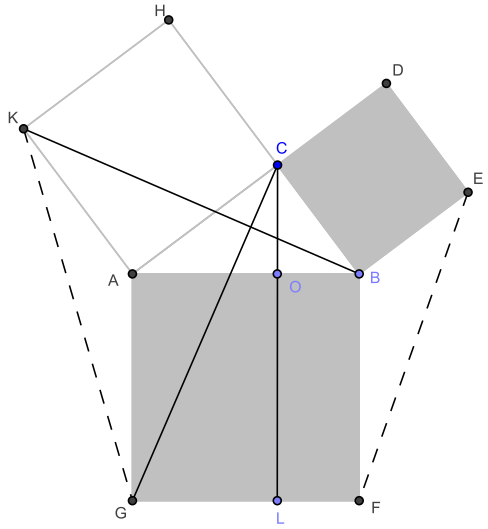


FIGURE 9.1.43. [1.47, #10]

We also have that

$$\begin{aligned} \text{Area } \triangle ABC &= \tfrac{1}{2}AB \cdot AC \cdot \sin(\angle BAC) \\ &= \tfrac{1}{2}AC \cdot AB \cdot \sin(\gamma) \\ &= \text{Area } \triangle KAG \end{aligned}$$

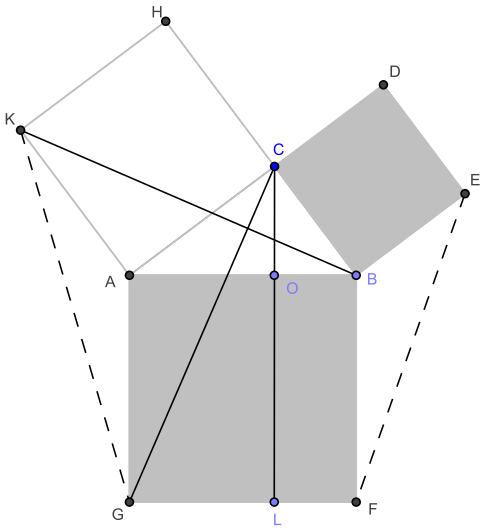
Similarly,

$$\begin{aligned} \text{Area } \triangle BEF &= \tfrac{1}{2}BE \cdot BF \cdot \sin(\angle FBE) \\ &= \tfrac{1}{2}BC \cdot AB \cdot \sin(\tfrac{\pi}{2} + \gamma) \\ &= \tfrac{1}{2}BC \cdot AB \cdot \cos(\gamma) \end{aligned}$$

and

$$\begin{aligned} \text{Area } \triangle ABC &= \tfrac{1}{2} AB \cdot BC \cdot \sin(\tfrac{\pi}{2} - \gamma) \\ &= \tfrac{1}{2} BC \cdot AB \cdot \cos(\gamma) \\ &= \text{Area } \triangle BEF \end{aligned}$$

Therefore $\triangle KAG = \triangle ABC = \triangle BEF$ which proves our claim. □



Chapter 1 exercises

1. Suppose $\triangle_1$ and $\triangle_2$ are triangles such that:

(a)$\triangle_1$ is constructed within $\triangle_2$

(b) each side of $\triangle_2$ passes through one vertex of $\triangle_1$

(c) each side of $\triangle_2$ is parallel to its opposite side in $\triangle_1$.

We claim that $\triangle_2 = 4 \cdot \triangle_1$.

PROOF. Construct:

1. $\triangle ABC$
2. $\overleftrightarrow{DF}$ such that A is a point on $\overleftrightarrow{DF}$ and $\overleftrightarrow{DF} \parallel \overline{BC}$
3. $\overleftrightarrow{EF}$ such that C is a point on $\overleftrightarrow{EF}$ and $\overleftrightarrow{EF} \parallel \overline{AB}$
4. $\overleftrightarrow{DE}$ such that B is a point on $\overleftrightarrow{DE}$ and $\overleftrightarrow{DE} \parallel \overline{AC}$.

We wish to show that the area of $\triangle DEF = 4 \cdot \triangle ABC$.

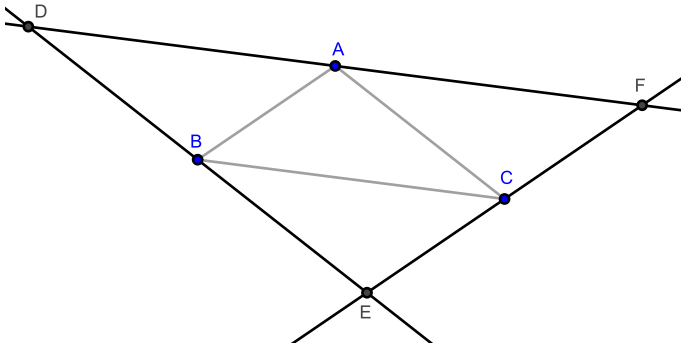


FIGURE 9.1.44. Chapter 1 exercises, #1

Since $\overleftrightarrow{DF} \parallel \overline{BC}$, $\angle ADB = \angle CBE$. Since $\overleftrightarrow{DE} \parallel \overline{AC}$, $\angle ADB = \angle CAF$. Hence, $\angle CBE = \angle CAF$

Similarly, we can show that $\angle CFA = \angle BAD = \angle ECB$ and $\angle DBA = \angle BEC = \angle ACF$.

Notice that $\boxdot AFCB$ is a parallelogram. Hence, $AF = BC$. Since $\boxdot DACB$ is also a parallelogram, $BC = DA$. By [1.26],

$$\triangle ABD \cong \triangle CEB \cong \triangle FCA$$

By [1.8],

$$\triangle ABD \cong \triangle CEB \cong \triangle FCA \cong \triangle ABC$$

which implies that their areas are equal. Since $\triangle DEF = \triangle ABD \oplus \triangle CEB \oplus \triangle FCA \oplus \triangle ABC$, the proof follows. $\square$

Chapter 1 exercises

4. The three medians of a triangle are concurrent. (Note: we are proving the existence of the **centroid** of a triangle.)

PROOF. Construct $\triangle ABC$ where $\overline{CE}$ is the median of AB and $\overline{BF}$ is the median of AC; let G be their point of intersection. Also construct $\overline{AD}$ such that G is a point of $\overline{AD}$. We claim that $\overline{AD}$ is the third median of $\triangle ABC$.

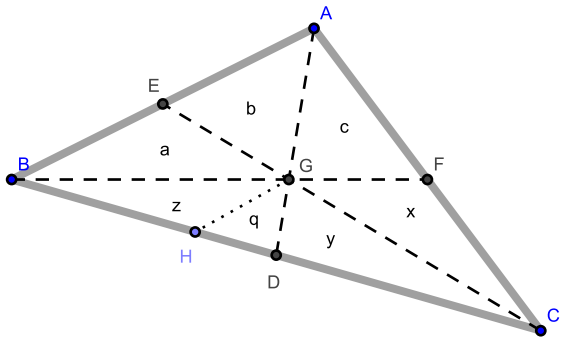


FIGURE 9.1.45. Chapter 1 exercises, #4

Construct $\overline{GH}$ and suppose that $\overline{BH} = \overline{HC}$ where $H \neq D$. Let the areas of each sub-triangle be written as the lowercase letters in the figure.

Consider $\triangle GBD$ and $\triangle GCD$. By [1.38],

$$z = q + y \tag{9.1.1}$$

Since $\overline{BF}$ is a median,

$$a + b + c = x + y + z + q \tag{9.1.2}$$

Since $\overline{CE}$ is a median,

$$x + b + c = y + z + q + a \tag{9.1.3}$$

Then (7.1.2)–(7.1.3) $= a - x = x - a$, or $x = a$. By [1.38], $a = b$ and $c = x$, so:

$$x = a = b = c$$

Then (7.1.2) can be written as $2x = y + z + q$.

Writing (7.1.1) as $0 = q + y - z$ and creating a system of equations with the above equation, we have:

$$\begin{aligned} 0 &= q + y - z \\ 2x &= q + y + z \\ &\Longrightarrow \\ x &= q + y \end{aligned}$$

It follows that $x = z$, or

$$x = a = b = c = z = q + y$$

and so

$$c + x + y + q = \frac{1}{2}\triangle ABC$$

Now construct $\overline{AH}$.

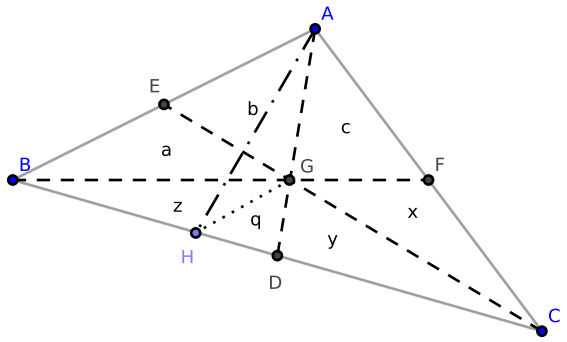


FIGURE 9.1.46. Chapter 1 exercises, #4

By [1.38, #1], every median bisects the triangle; since $\overline{BH} = \overline{HC}$, $\overline{AH}$ is a median. Notice that

$$c + x + y + q < \triangle ACH = \frac{1}{2} \cdot \triangle ABC$$

a contradiction.

Hence, $\overline{BD} = \overline{DC}$ and $\overline{AD}$ is the third median. □

COROLLARY. *The three medians of a triangle divide the original triangle into six sub-triangles which are each equal in area.*

Chapter 1 exercises

8. Construct a triangle given the three medians.

PROOF. Suppose we are given the medians of a triangle: $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$. We shall construct $\triangle ABC$.

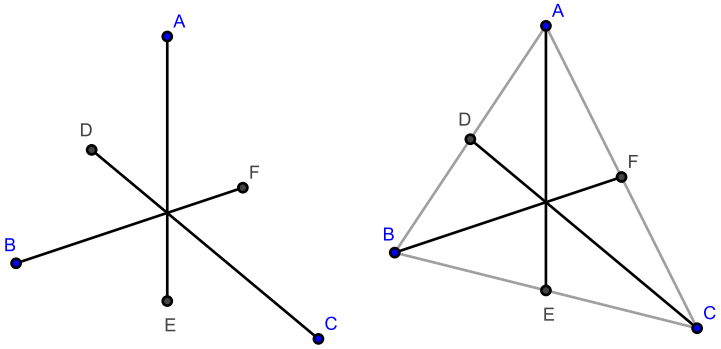


FIGURE 9.1.47. Chapter 1 exercise #8

Construct $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. Since $\overline{AE}$ is a median, $\overline{BE} \oplus \overline{EC} = \overline{BC}$ where $\overline{BE} + \overline{EC}$. Similar statements can be made for the remaining sides, *mutatis mutandis*. Hence, $\triangle ABC$ is constructed. □

Chapter 1 exercises

12. The shortest median of a triangle corresponds to the largest side.

PROOF. Construct $\triangle ABC$ where $AB < AC < BC$, $\overline{CE}$ is the median of AB, $\overline{BF}$ is the median of AC, and $\overline{AD}$ is the median of BC. We wish to show that $\overline{AD} < \overline{BF} < \overline{CE}$.

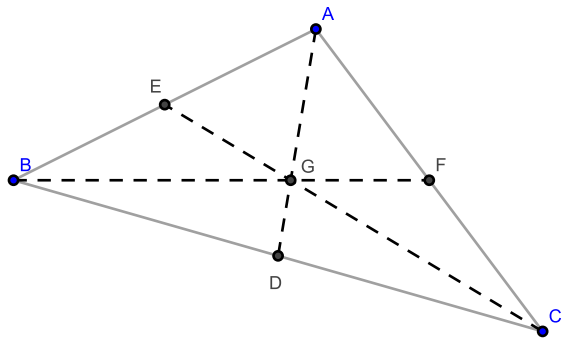


FIGURE 9.1.48. Chapter 1 exercises, #12

Consider $\triangle BGD$ and $\triangle BGE$. By the Corollary to [Ch.1 Exercises, #4], we have that $\triangle BGD = \triangle BGE$. Since $AB < BC$ by hypothesis and $\overline{CE}$, $\overline{AD}$ are medians,

$$BE = \frac{1}{2} \cdot AB < \frac{1}{2} \cdot BC = BD$$

Since $\triangle BDG$ and $\triangle BGE$ share side BG, we must have that $\overline{DG} < \overline{EG}$ (otherwise the areas of these triangles would not be equal).

Similarly, it can be shown that $\overline{DG} < \overline{FG} < \overline{EG}$. By [Ch. 6 Exercises. #124], we have that the medians of a triangle divide each other in the ratio of $2 : 1$. Hence, $\overline{AD} < \overline{FB} < \overline{EC}$. □

Chapter 1 exercises

16. Inscribe a rhombus in a triangle having for an angle one angle of the triangle.

PROOF. Construct $\triangle ABC$. Let $\angle DAB$ bisect $\angle CAB$ where D is a point on the side BC. Construct the ray $\overrightarrow{AD}$. Construct $\angle ADF$ and $\angle ADE$ such that

$$\angle ADF = \angle DAB = \angle ADE$$

We claim that $\boxdot AEDF$ is the required rhombus.

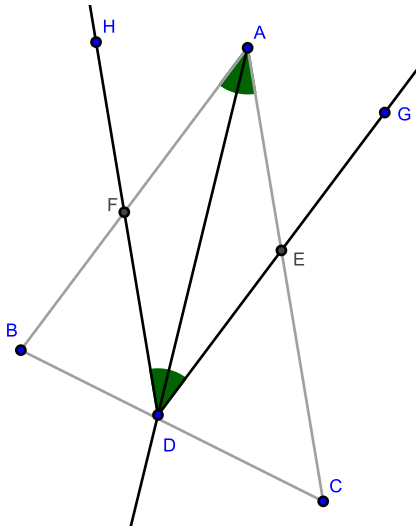


FIGURE 9.1.49. [Ch. 1 Exercises, #16]

Consider $\triangle DFA$ and $\triangle DEA$: $AF = AE$, $\angle DAF = \angle DAE$ by construction, and the triangles share side AD. By [1.4], $\triangle DFA \cong \triangle DEA$. Hence,

$$DF = FA = AE = ED$$

By [Def. 1.29], $\boxdot AEDF$ is a rhombus. Since $\boxdot AEDF$ is in $\triangle ABC$ and $\triangle ABC$ shares $\angle BAC$ with $\boxdot AEDF$, our claim is proven. $\square$

9.2. Solutions for Chapter 2

[2.4] Exercises

2. If from the right angle of a right triangle a perpendicular falls on the hypotenuse, its square equals the area of the rectangle contained by the segments of the hypotenuse.

PROOF. Construct right triangle $\triangle ABC$ where $\angle BAC$ is a right angle. Construct segment $\overline{AD}$ such that $\overline{AD} \perp \overline{BC}$. We claim that $(\overline{AD})^2 = \overline{DB} \cdot \overline{DC}$.

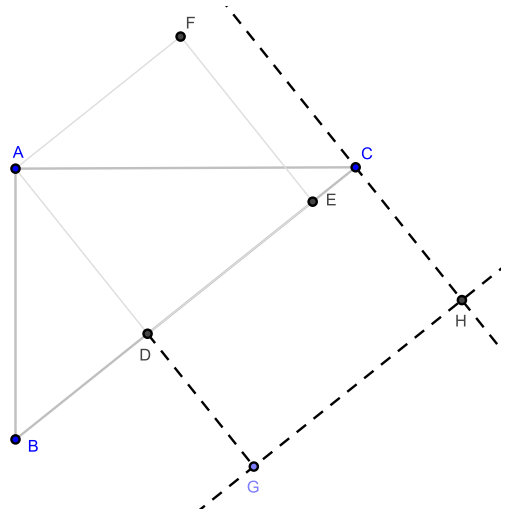


FIGURE 9.2.1. [2.4, #2]

Construct rectangle $\boxdot DCHG$ where $\overline{BD} = \overline{DG}$. (Note: geometrically, we claim that $(\overline{AD})^2 = \boxdot DCHG$.)

By [1.47], we have that

$$\begin{aligned} (\overline{AD})^2 + (\overline{DC})^2 &= (\overline{AC})^2 \\ (\overline{AD})^2 + (\overline{DB})^2 &= (\overline{AB})^2 \end{aligned}$$

as well as

$$\begin{aligned} (\overline{AB})^2 + (\overline{AC})^2 &= (\overline{DB} + \overline{DC})^2 \\ &= (\overline{DB})^2 + 2 \cdot \overline{DB} \cdot \overline{DC} + (\overline{DC})^2 \end{aligned}$$

Hence,

$$\begin{aligned}
(\overline{AD})^2 + (\overline{DC})^2 + (\overline{AD})^2 + (\overline{DB})^2 &= (\overline{AB})^2 + (\overline{AC})^2 \\
2 \cdot (\overline{AD})^2 + (\overline{DC})^2 + (\overline{DB})^2 &= (\overline{DB})^2 + 2 \cdot \overline{DB} \cdot \overline{DC} + (\overline{DC})^2 \\
2 \cdot (\overline{AD})^2 &= 2 \cdot \overline{DB} \cdot \overline{DC} \\
(\overline{AD})^2 &= \overline{DB} \cdot \overline{DC}
\end{aligned}$$

This proves our claim. □

[2.6] Exercises

7. Give a common statement which will include [2.5] and [2.6].

PROOF. Construct $\overleftrightarrow{AB}$ and consider the segment formed by points A and B. Locate the midpoint C of this segment. Choose a point D on $\overleftrightarrow{AB}$ such that D is neither A, B, nor C. We then have two cases:

1) D is between A and B. By [2.5], we have that

$$\overline{AD} \cdot \overline{DB} + (\overline{CD})^2 = (\overline{CB})^2$$

2) D is not between A and B. By [2.6], we have that

$$\overline{AD} \cdot \overline{DB} + (\overline{CB})^2 = (\overline{CD})^2$$

□

[2.11] Exercises

3. If $\overline{AB}$ is cut in "extreme and mean ratio" at C, prove that

(a) $(\overline{AB})^2 + (\overline{BC})^2 = 3 \cdot (\overline{AC})^2$

PROOF. By [2.11], $x = -\frac{a}{2}(1 \pm \sqrt{5})$. (We may ignore negative results since our context is the side-length of planar figures.) Since $\overline{AB} = a$, $\overline{BC} = a - x$, and $\overline{AC} = x$, we have:

$$\begin{aligned}
(\overline{AB})^2 + (\overline{BC})^2 &= 3 \cdot (\overline{AC})^2 \\
a^2 + (a - x)^2 &= 3x^2 \\
a^2 + a^2 - 2ax + x^2 &= 3x^2 \\
2a^2 - 2ax &= 2x^2 \\
a^2 - ax &= x^2 \\
a^2 - \tfrac{a^2}{2}(1 + \sqrt{5}) &= \tfrac{a^2}{2}(1 + \sqrt{5}) \\
\tfrac{a^2}{2}(1 + \sqrt{5}) &= \tfrac{a^2}{2}(1 + \sqrt{5})
\end{aligned}$$

□

Chapter 2 exercises

15. Any rectangle is equal in area to half the rectangle contained by the diagonals of squares constructed on its adjacent sides.

PROOF. Construct rectangle $\boxdot ADCB$, squares $\boxdot GABH$ and $\boxdot BCFE$, and diagonals $\overline{GB}$ and $\overline{BF}$. We claim that $\frac{1}{2}(\overline{GB})^2 \cdot (\overline{BF})^2 = \overline{AB} \cdot \overline{BC}$.

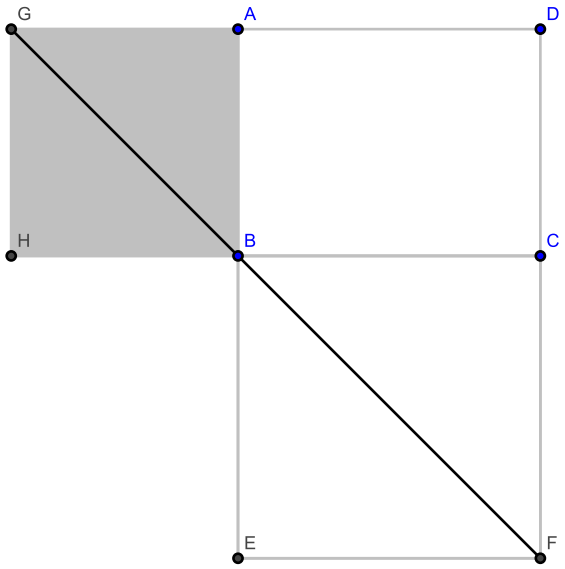


FIGURE 9.2.2. [Ch. 2 Exercises, #15]

Let $\overline{AB} = x$ and $\overline{BC} = y$. By [1.47], it follows that $\overline{GB} = x\sqrt{2}$ and $\overline{BC} = y\sqrt{2}$. Notice that

$$\begin{aligned} \frac{1}{2}(\overline{GB})^2 \cdot (\overline{BF})^2 &= \frac{1}{2}xy\sqrt{4} \\ &= xy \\ &= \overline{AB} \cdot \overline{BC} \end{aligned}$$

□

9.3. Solutions for Chapter 3

[3.3] Exercises

5. Prove [3.3, Cor. 4]: The line joining the centers of two intersecting circles bisects their common chord perpendicularly.

PROOF. Construct the circles from [1.1], and then construct segment $\overline{CF}$. We claim that $\overline{AB}$ bisects $\overline{CF}$ and that $\overline{AB} \perp \overline{CF}$.

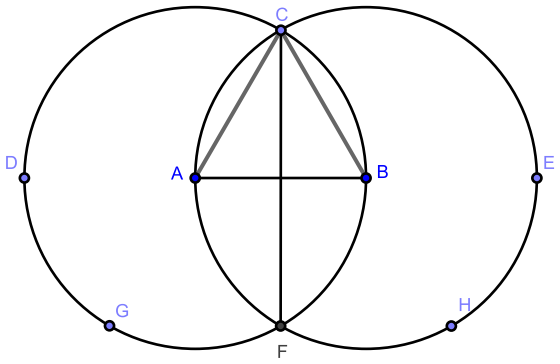


FIGURE 9.3.1. [3.3, #3]

The claim follows immediately from the proof of [1.1, #2]. The details are left as an exercise to the reader. □

[3.5] Exercises

2. Two circles cannot have three points in common without coinciding.

PROOF. Suppose instead that two circles ($\bigcirc C$, $\bigcirc A$) have three points in common (E, F, and G) and do not coincide.

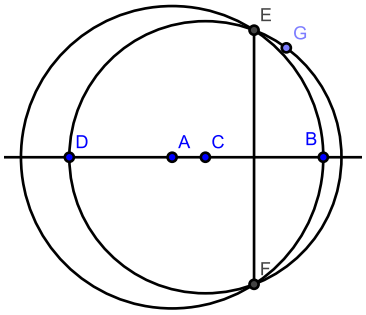


FIGURE 9.3.2. [3.5, #2]

By [3.3, Cor. 4], the line joining the centers of two intersecting circles ($\overleftrightarrow{AC}$) bisects their common chord perpendicularly; hence, $\overleftrightarrow{AC} \perp \overline{EF}$.

Similarly, $\overleftrightarrow{AC}$ bisects $\overline{EG}$. But $\overline{EG}$ can be constructed so that $\overline{EG}$ and $\overleftrightarrow{AC}$ do not intersect, a contradiction. Therefore, $\bigcirc C$ and $\bigcirc A$ coincide. □

[3.13] Exercises

3. Suppose two circles touch externally. If through the point of intersection any secant is constructed cutting the circles again at two points, the radii constructed to these points are parallel.

PROOF. Suppose $\bigcirc A$ and $\bigcirc C$ touch at point B. By [3.13], these circles touch only at B. Construct secant $\overline{DBE}$. We claim that $\overline{AD} \parallel \overline{CE}$.

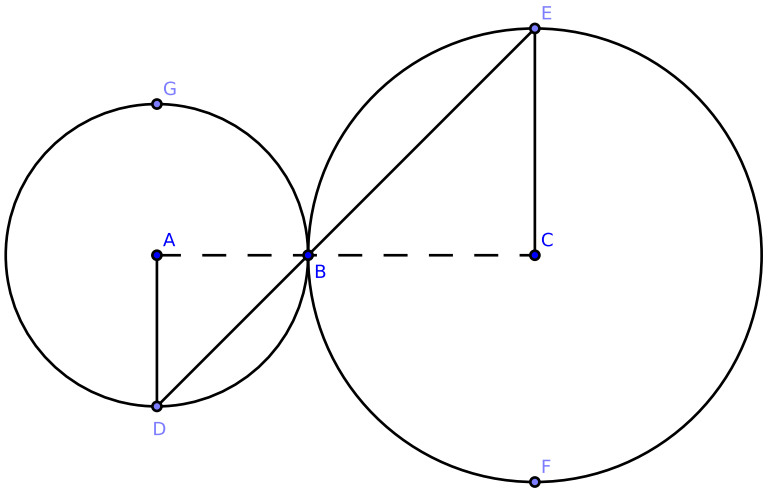


FIGURE 9.3.3. [3.13, #3]

Connect $\overline{AC}$. By [3.12], $\overline{AC}$ intersects B.

Consider $\triangle ABD$ and $\triangle CBE$: $\angle ABD = \angle CBE$ by [1.15]; since each triangle is isosceles, $\angle ADB = \angle CEB$. By [1.29, Cor. 1], $\overline{AD} \parallel \overline{CE}$. This proves our claim. □

COROLLARY. *If two circles touch externally and through the point of intersection any secant is constructed cutting the circles again at two points, the diameters constructed to these points are parallel.*

[3.13] Exercises

4. Suppose two circles touch externally. If two diameters in these circles are parallel, the line from the point of intersection to the endpoint of one diameter passes through the endpoint of the other.

PROOF. Suppose $\bigcirc A$ and $\bigcirc C$ touch at point B. By [3.13], these circles touch only at B. Construct $\overline{AC}$, and construct diameters $\overline{DE}$ and $\overline{FG}$ such that $\overline{DE} \parallel \overline{FG}$ and $\overline{DE} \perp \overline{AC}$. It follows that $\overline{AC} \perp FG$.

We claim that $\overleftrightarrow{DB}$ intersects G.

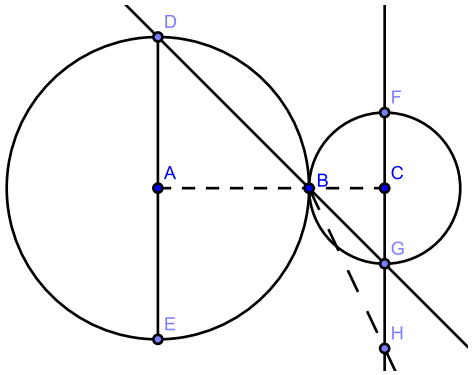


FIGURE 9.3.4. [3.13, #4]

Suppose that $\overleftrightarrow{DB}$ does not intersect G. Extend $\overline{FG}$ to $\overleftrightarrow{FG}$ and suppose that $\overleftrightarrow{DB}$ intersects $\overleftrightarrow{FG}$ at H.

Consider $\triangle ABD$ and $\triangle CBH$: by hypothesis, $\angle DAB = \angle HCB$; by [1.15], $\angle ABD = \angle CBH$; by [1.29, Cor. 1], $\angle ADB = \angle CHB$. Or, $\triangle ABD$ and $\triangle CBH$ are equiangular.

However, $\triangle ABD$ is an isosceles triangle and $\triangle CBH$ is not since $\overline{BC} = \overline{CG}$ and $\overline{CG} < \overline{CH}$; hence, the triangles are not equiangular, a contradiction.

An equivalent contradiction is obtained if $\overline{DB}$ intersects the line $\overline{FG}$ at any point other than G, *mutatis mutandis*. This proves our claim.

□

[3.16] Exercises

1. If two circles are concentric, all chords of the greater circle which touch the lesser circle are equal in length.

PROOF. Construct $\bigcirc A_1$ with radius $\overline{AB}$ and $\bigcirc A_2$ with radius $\overline{AC}$. On $\bigcirc A_2$, construct chord $\overline{DE}$ such that $\overline{DE}$ touches $\bigcirc A_1$ at B. Also on $\bigcirc A_2$, construct chord $\overline{HJ}$ such that $\overline{HJ}$ touches $\bigcirc A_1$ at G. We claim that $\overline{DE} = \overline{HJ}$.

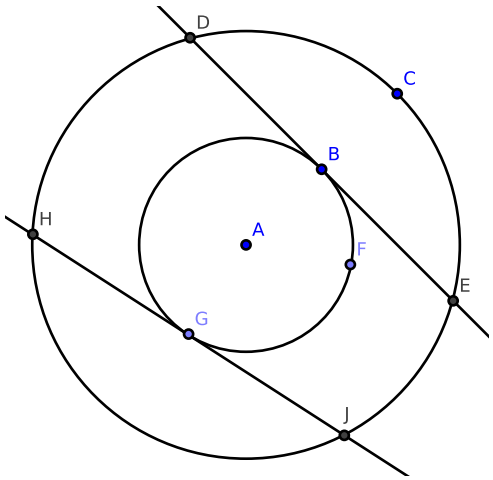


FIGURE 9.3.5. [3.16, #1]

Notice that $\overline{AG} = \overline{AB}$ since each are radii of $\bigcirc A_1$. By [3.16], $\overline{DE}$ and $\overline{HJ}$ have no other points of intersection with $\bigcirc A_1$. Hence, $\overline{DE}$ and $\overline{HJ}$ are equal distance from A, the center of $\bigcirc A_2$. By [3.14], $\overline{DE} = \overline{HJ}$. This proves our claim. □

[3.17] Exercises

3. If a parallelogram is circumscribed to a circle, it must be a rhombus, and its diagonals intersect at the center.

PROOF. Construct circle $\bigcirc E$ and circumscribe parallelogram $\boxdot BCDA$ to $\bigcirc E$ where $\boxdot BCDA$ touches $\bigcirc E$ at F, G, H, and I. We claim that $\boxdot BCDA$ is a rhombus and that its diagonals $\overline{AC}$, $\overline{BC}$ intersect at the center of $\bigcirc E$.

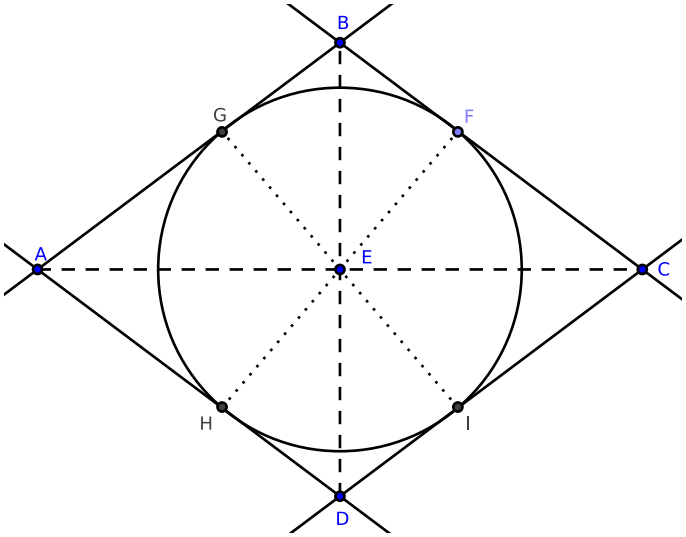


FIGURE 9.3.6. [3.17, #3]

Since $\boxdot BCDA$ is a parallelogram, $\overline{EA} = \overline{EC}$, $\overline{EB} = \overline{ED}$, $\overline{BC} = \overline{AD}$ and $\overline{BA} = \overline{CD}$. By [1.8], $\triangle EAD \cong \triangle ECB$ and $\triangle EBA \cong \triangle EDC$.

Construct radii $\overline{EH}$, $\overline{EI}$. Since $\bigcirc E$ touches $\boxdot BCDA$ at H and at I, we must have and that $\angle EHD = \angle EID$, since each are right angles; therefore, $\triangle EHD$ and $\triangle EID$ are right triangles.

Consider $\triangle EHD$ and $\triangle EID$: each shares side $\overline{ED}$, $\overline{EH} = \overline{EI}$ (since both are radii of $\bigcirc E$), and so by [1.26, #6] we have that $\triangle EHD \cong \triangle EID$. Similarly, $\triangle EHA \cong \triangle EIC$, and so it follows that $\triangle EAD \cong \triangle ECD$, and

$$\overline{AD} = \overline{CD} = \overline{BA} = \overline{BC}$$

and so $\boxdot BCDA$ is a rhombus.

Notice that three or more equal segments can be constructed from E to the circumference of the circle. By [3.9], E is the center of the circle, and so the diagonals of $\boxdot BCDA$ intersect at the center of $\bigcirc E$. $\square$

[3.30] Exercises

1. Suppose that $ABCD$ is a semicircle whose diameter is $\overline{AD}$ and with chord $\overline{BC}$. Extend $\overline{BC}$ to $\overrightarrow{BC}$ and $\overline{AD}$ to $\overrightarrow{AD}$, and suppose each ray intersects at E. Prove that if $\overline{CE}$ is equal in length to the radius, the arc $\overline{AB} = 3 \cdot \overline{CD}$.

PROOF. Our hypothesis and claim are stated above.

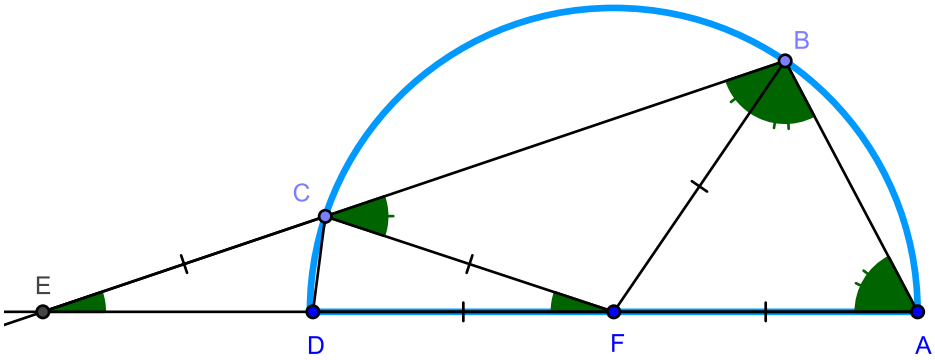


FIGURE 9.3.7. [3.30, #1]

Construct $\overline{CD}$, $\overline{CF}$, $\overline{FB}$, and $\overline{AB}$. Notice that

$$\overline{CE} = \overline{DF} = \overline{CF} = \overline{BF} = \overline{AF}$$

By [1.5], we have that $\angle CEF = \angle CFE$, $\angle FCB = \angle FBC$, and $\angle FBA = \angle FAB$.

Using linear algebra, let

$$\begin{aligned} \angle CEF &= a \\ \angle ECF &= b \\ \angle FCB &= c \\ \angle CFB &= d \\ \angle FBA &= e \\ \angle BFA &= f \end{aligned}$$

Then we have that

$$\begin{aligned} b + c &= 180 \\ 2a + b &= 180 \\ 2c + d &= 180 \\ 2e + f &= 180 \\ a + d + f &= 180 \\ a + c + 2e &= 180 \end{aligned}$$

In matrix form, we obtain:

$$\left[\begin{array}{cccccc|c} 0 & 1 & 1 & 0 & 0 & 0 & 180 \\ 2 & 1 & 0 & 0 & 0 & 0 & 180 \\ 0 & 0 & 2 & 1 & 0 & 0 & 180 \\ 0 & 0 & 0 & 0 & 2 & 1 & 180 \\ 1 & 0 & 0 & 1 & 0 & 1 & 180 \\ 1 & 0 & 1 & 0 & 2 & 0 & 180 \end{array}\right]$$

In reduced row echelon form, we obtain:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 0 & 0 & 140 \\ 0 & 0 & 1 & 0 & 0 & 0 & 40 \\ 0 & 0 & 0 & 1 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 60 \\ 0 & 0 & 0 & 0 & 0 & 1 & 60 \end{array}\right]$$

Or,

$$\begin{aligned} \angle CEF &= 20^\circ \\ \angle ECF &= 140^\circ \\ \angle FCB &= 40^\circ \\ \angle CFB &= 100^\circ \\ \angle FBA &= 60^\circ \\ \angle BFA &= 60^\circ \end{aligned}$$

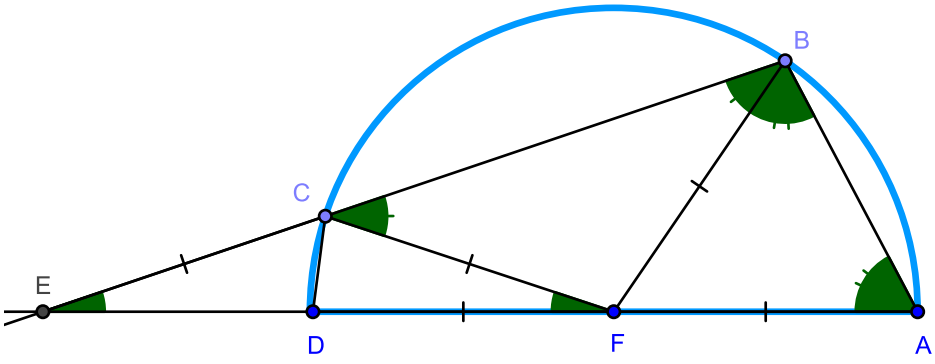


FIGURE 9.3.8. [3.30, #1]

Since $\angle CFD = \angle CFE = \angle CEF$, $\angle BFA = 3 \cdot \angle CFD$. By [7.29, Cor. 1], it follows that $\overline{AB} = 3 \cdot \overline{CD}$. □

9.4. Solutions for Chapter 4

[4.4] Exercises

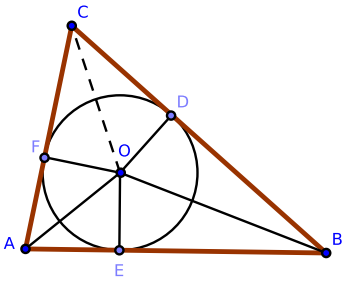


FIGURE 9.4.1. [4.4, #1]

1. In [4.4]: if $\overline{OC}$ is constructed, prove that the angle $\angle ACB$ is bisected. Hence, we prove the existence of the *incenter* of a triangle.

PROOF. Consider $\triangle OFC$ and $\triangle CDO$. By the proof of [4.4], we have that $\angle OFC = \angle CDO$ since each are right; $\overline{OF} = \overline{OD}$ since each are radii of $\bigcirc O$; and each shares side $\overline{OC}$. By [1.26, #6], $\triangle OFC \cong \triangle CDO$, and so $\angle OCF = \angle CDO$. Since $\angle ACB = \angle OCF \oplus \angle CDO$, $\angle ACB$ is bisected, and therefore O is the incenter of $\triangle ABC$. $\square$

[4.5] Exercises

1. Prove that the three altitudes of a triangle ($\triangle ABC$) are concurrent. (This proves the existence of the *orthocenter* of a triangle.)

PROOF. Euler's proof: consider $\triangle ABC$ with circumcenter O and centroid G (i.e., the point of intersection of the medians of the triangle).

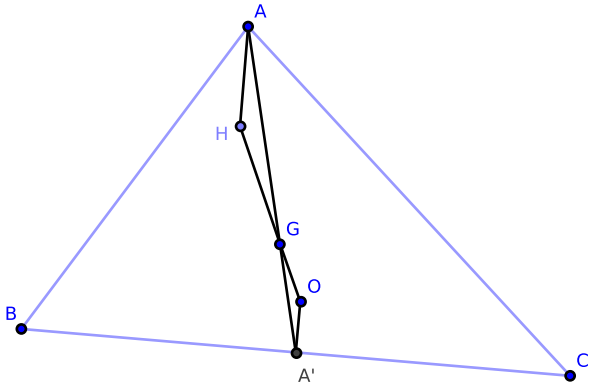


FIGURE 9.4.2. [4.5, #1]

Let A' be the midpoint of $\overline{BC}$, and let H be the point such that G is between H and O and $\overline{HG} = 2 \cdot \overline{GO}$.

Then the triangles $\triangle AGH$, $\triangle A'GO$ are similar by side-angle-side similarity. It follows that $\overline{AH} \parallel \overline{OA'}$ and therefore $\overline{AH} \perp \overline{BC}$; i.e., it is the altitude from A.

Similarly, $\overline{BH}$ and $\overline{CH}$ are the altitudes from B and C, respectively. Hence all the altitudes pass through H.[1] □

[1] http://www.artofproblemsolving.com/Wiki/index.php/Orthocenter

[4.7] Exercises

1. Prove [Cor. 4.7.1]: the circumscribed square, $\boxdot EFGH$, has double the area of the inscribed square, $\boxdot BCDA$.

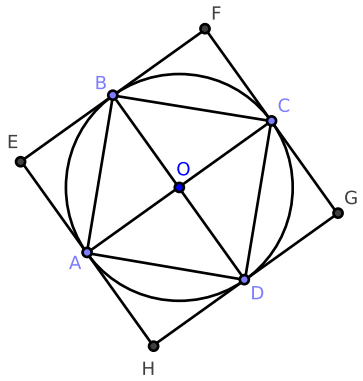


FIGURE 9.4.3. [4.7]

PROOF. Consider $\boxdot EBOA$: $\overline{BA}$ is a diagonal, so $\triangle BOA = \frac{1}{2} \cdot \boxdot EBOA$ by [1.34].

Since

$$\boxdot EBOA = \boxdot BFCO = \boxdot OCGD = \boxdot ODHA$$

by the proof of [4.7], we have that

$$\triangle BOA = \triangle BOC = \triangle CDO = \triangle AOD$$

Notice that

$$\boxdot BCDA \quad = \quad \triangle BOA \oplus \triangle BOC \oplus \triangle CDO \oplus \triangle AOD$$

and so

$$\boxdot BCDA \quad = \quad \frac{1}{2} \cdot (\boxdot EBOA + \boxdot BFCO + \boxdot OCGD + \boxdot ODHA)$$

Since

$$\boxdot EFGH = \boxdot EBOA + \boxdot BFCO + \boxdot OCGD + \boxdot ODHA$$

we have that

$$\boxdot EFGH = 2 \cdot \boxdot BCDA$$

This proves our claim. □

[4.10] Exercises

1. Prove that $\triangle ACD$ is an isosceles triangle whose vertical angle is equal to three times each of the base angles.

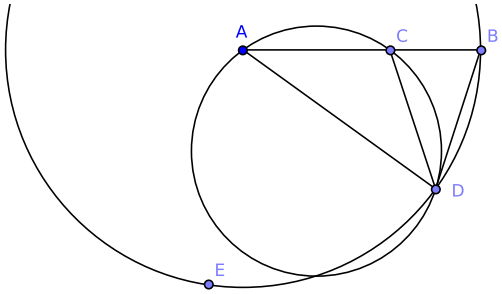


FIGURE 9.4.4. [4.10]

PROOF. We claim that $\triangle ACD$ is isosceles where

$$\angle ACD = 3 \cdot \angle DAC = 3 \cdot \angle ADC$$

Since $\angle BDA = \angle DBA = 2 \cdot \angle DAB$ by [4.10], we must have that

$$\begin{aligned} \angle DAB &= 36^\circ \\ \angle DBA &= 72^\circ \\ \angle BDA &= 72^\circ \end{aligned}$$

By the proof of [4.10], $\overline{BD} = \overline{CD}$; by [Cor. 1.6.1], $\angle DBC = \angle DCB$. By [1.13], $\angle ACD = 108^\circ$. Since $\angle DAB = \angle DAC = 36^\circ$, it follows that $\angle ADC = 36^\circ$.

Then $\triangle ACD$ is isosceles since $\angle DAC = \angle ADC$. And since $36 \cdot 3 = 108$, our claim is proven. □

9.5. Solutions for Chapter 5

Chapter 5 Exercises

6. If four numbers are proportionals, their squares, cubes, etc., are proportionals.

PROOF. Let a, b, x, and y be positive integers such that

$$\frac{a}{b} = \frac{x}{y}$$

We claim that

$$\left(\frac{a}{b}\right)^n = \left(\frac{x}{y}\right)^n$$

where $n \geq 1$ is a positive integer.

The equality holds for $n = 1$ by assumption. Assume that the equality holds for $n = k$:

$$\left(\frac{a}{b}\right)^k = \left(\frac{x}{y}\right)^k$$

Then

$$\begin{aligned}
\left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right)^k &= \left(\frac{a}{b}\right) \cdot \left(\frac{x}{y}\right)^k \\
\left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right)^k &= \left(\frac{x}{y}\right) \cdot \left(\frac{x}{y}\right)^k \\
\left(\frac{a}{b}\right)^{k+1} &= \left(\frac{x}{y}\right)^{k+1}
\end{aligned}$$

And so the equality holds for $n = k + 1$, and therefore it holds for $n \geq 1$. The proves our claim. □

Questions? Comments? Did you find an error?

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Make sure to include the version number, which can be found on page 2 of this document.